

The Cauchy Horizon In Black Hole-de Sitter Spacetimes

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The last seven years has produced a growing body of evidence which concludes that the Cauchy horizon in black hole de Sitter spacetimes is classically stable when the surface gravity at the cosmological event horizon is greater than that at the Cauchy horizon. That stability persists for a finite, but non-zero, region of the black hole's parameter space, (M, Q, J, Λ) , suggests that black holes immersed in de Sitter space are counter-examples to the strong cosmic censorship hypothesis.

In this review we chronicle that body of evidence and describe the first steps of a program of numerical work aimed at better understanding the interior of black hole-de Sitter spacetimes. The review ends with a speculative account of the role that future work will take.

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I. INTRODUCTION

Our discussions at Haifa have, among other things, emphasized the need for a correct formulation and proof of the *Strong Cosmic Censorship Hypothesis*. Stated in its simplest, physical form, the strong cosmic censorship hypothesis pronounces;

All physically reasonable spacetimes are globally hyperbolic – Apart from a possible initial singularity, no spacetime singularity is ever visible to any observer.

Unfortunately cosmic censorship is an area which yields little return on much effort. We believe, however, we shall eventually be guided to a precise formulation and proof by examining example and counter-example to the hypothesis. While counter-examples are abundant in the literature, few can live up the claim of being ‘reasonable’ spacetimes. There are a few notable exceptions to this rule though, in particular the Ernst spacetime, a solution of the Einstein-Maxwell equations [1] and the focus of this contribution, black holes immersed in de Sitter space. The familiarity of black hole solutions make the de Sitter example a particularly attractive field of study.

Black Holes In de Sitter Space

The fate of an observer falling in to a black hole is both an interesting and important problem within the framework of classical general relativity. The issue takes on a more notable significance if the black hole is charged or rotating. In this case the observer’s journey appears to continue through the interior of the black hole and beyond, eventually emerging from a white hole in to a new universe. The simplicity of this picture, however, belies the underlying physics. Deep within the black hole, concealed from the exterior by an event horizon, resides a spacetime singularity, characterized by infinite curvatures, and more importantly for our observer, infinite tidal forces. While fully able to avoid the crushing grip of the singularity, due in part to its timelike nature, the observer is unable to prevent the blatant violation of strong cosmic censorship that such a journey irrevocably promises – a sighting of the singularity. If the issues concerning how physics is going to describe such a *naked* singularity weren’t enough, our observer is faced with an even more fundamental, though related, problem – a loss of predictability. It is well known that general relativity admits a well posed initial value problem, i.e., given some suitable initial data on a spacelike hypersurface, S say, the solution to the Einstein equations is uniquely determined everywhere within the domain of dependence $D(S)$ of S [2]. For some spacetimes though, $D(S)$ can often fail to cover the entire manifold M . So, even with suitable initial data on S , general relativity is unable to forecast the evolution of the spacetime beyond $D(S)$ in these cases. The boundary of $D(S)$, $H(S)$, is called the Cauchy horizon and marks the division between the region where general relativity is able to predict the evolution and the region where predictability of the field equations is lost. Black holes with charge or rotation, such as the Reissner-Nordström and Kerr solutions, are well known to possess such an horizon. For these black holes the Cauchy horizon coincides with an inner horizon that veils the singularity. In crossing the event horizon, all observer have unwittingly committed themselves to a journey that will inevitably encounter and attempt to cross this curious frontier that is the Cauchy horizon – an expedition that must contend with naked singularities, loss of predictability and, in some cases, causality violation. It may come as some relief to hear that physics (or nature if you prefer) appears to abhor this situation too and, indeed, conspires to prevent the passage through the interior, terminating it at the Cauchy horizon [3]. The conspiracy itself is rooted in an instability of the inner horizon to even the smallest time dependent perturbation, converting an initially regular horizon in to a null, spacetime curvature singularity, effectively sealing off the tunnel to other worlds. The source of the instability can be understood quite simply as an *infinite proper time compression* effect. Because the causal past of the Cauchy horizon contains the entire universe external to the black hole, any observer approaching the horizon sees an infinite number of events in a finite proper time. Stated more dramatically; the observer sees the entire history of the external universe flash before their eyes, in the last few moments before they cross the Cauchy horizon. That this compression effect becomes larger as one approaches the Cauchy horizon, results in progressively larger energy densities being measured, until, at the horizon itself, they become infinite, and turn it in to a spacetime singularity. Thus the Cauchy horizon instability restores predictability to the situation and provides an excellent example of a strong cosmic censorship obeying spacetime. So what if the causal past of the Cauchy horizon does not contain the entire external universe but just some part of that universe? In that case one can easily conceive a scenario in which the observer crossing the Cauchy horizon sees only a finite number of events in a finite proper time, possibly leading to no infinite blueshift effects and hence no infinite energy densities or curvature singularities at the horizon. In effect, such a spacetime could have a Cauchy horizon that is stable to time dependent perturbations and serve as a counter-example to strong cosmic censorship (reasonable or unreasonable). Does nature, in an attempt to prevent the embarrassment of naked singularities, plot to prevent such a spacetime occurring in a physically realistic situation? Amazingly it appears not.

As we mentioned earlier, black holes immersed in de Sitter space prove to be notable counter-examples, for de Sitter black holes are part of a closed universe and play out much of the scenario discussed above. The familiarity of black hole solutions, combined with the knowledge that all the known black hole spacetimes of the Kerr-Newman family can be generalized to include a cosmological constant, establishes black hole de Sitter spacetimes as a favorable field of investigation.

During the last seven years a consistent picture of a classically stable Cauchy horizon in black hole-de Sitter spacetimes has emerged. Both linear studies and non-linear backreaction calculations indicate that the Cauchy horizon is stable. Although the scenario envisaged above, based on the infinite time compression effects, suggests the Cauchy horizon will always be stable, all analyses agree that stability persists for only a finite, but non-zero, measure on the parameter space (M, Q, J, Λ) of the black hole, infringing on the very spirit of the strong cosmic censorship hypothesis. Whether one views these spacetimes as reasonable counter-examples to the strong cosmic censorship hypothesis is a matter for personal tastes. In the past, much observational evidence was advanced to suggest that the cosmological constant, made famous by Einstein, is zero. Lately (within the last year), new evidence, based on improved gravitational lensing and cosmic microwave background observations, has surfaced and is currently challenging the established view of a vanishing cosmological constant [4]. At the least, since a cosmological constant does not violate any known law of physics, black hole-de Sitter spacetimes provide an excellent arena in which to examine the current form of the strong cosmic censorship hypothesis. The appeal of black hole-de Sitter spacetimes is also realized when comparing them with their asymptotically flat counterparts. While the Cauchy horizon in the Reissner-Nordström spacetime is well known to be unstable it is intriguing that the metric, expressed in well behaved coordinates, is regular there. Further to this, it has been demonstrated that while the tidal forces, due to the strong gravitational field of the Cauchy horizon singularity, on an observer grow without bound as the Cauchy horizon is approached the actual tidal deformation suffered by an observer does not grow without bound. These facts have re-opened the issues of traversability of the Cauchy horizon, and formed a major part of the discussions at Haifa. The answers to this debate are by no means clear. It has been proposed that the finite tidal deformations are irrelevant to the question of traversability [5] – any infalling object is completely destroyed at the Cauchy horizon since the energy absorbed by the object diverges as the horizon is approached. Our black hole-de Sitter model has none of these interpretational problems [6]. In this respect, the dilemmas that a traversible Cauchy horizon faces us with, are more clear cut in the case of black hole-de Sitter spacetimes than they are in the asymptotically flat case.

Arrangement

In accordance with the requirements of the organizers of the “Internal Structure of Black Holes and Spacetime Singularities” meeting at Haifa, this contribution has been written at a level and conciseness specifically intended for those who are conversant with General Relativity but are not familiar with the subject of Cauchy horizon stability, particularly graduate students for whom the subject literature, and logic behind it, is often confusing. That said, it is also hoped that those working in this and related fields will find this a worthy offering to the subject area. While every attempt has been made to present an impartial account of the subject, the narration undoubtedly suffers from my own personal bias.

In section II we present a mathematical description of black hole-de Sitter spacetimes. The section attempts to provide most of the basic mathematical material that is required for an understanding of later sections. Where this is not been possible the reader has been guided to compensating reference material. For the sake of simplicity, attention is focused exclusively on the spherically symmetric Reissner-Nordström-de Sitter solution. Much of what needs to be known about black hole-de Sitter spacetimes is most easily gained through studying the Reissner-Nordström-de Sitter spacetime. The different coordinate systems used throughout the spacetime, and the role they play, is explained and emphasized with the section concluding on a study of the behavior of radially free-falling geodesic observers. While the purpose of some of the concepts may not be immediately apparent, they will become more clear on a second read through.

Section III chronicles an account of the relevant works performed to date. While each commentary is not exhaustive, every attempt has been made to produce a clear and concise summary of that particular study. Specific emphasis has been placed on the motivation and philosophy of the approach and a precise statement of the results obtained. Each account is followed by critical review of the method, and results, in an endeavor to provide the continuity required to go from one study to the next and to illustrate the logic in that step.

In Sec. IV we provide additional material, in the form of comments, that have not, hereto, been published. Much of this material originates from the questions and comments that have arisen at previous conferences and from the many personal discussions that occurred during the Haifa meeting. With the knowledge of Sec. III at hand, these questions are explained and an attempt to address them is made. The conclusions of this section leads us nicely on

to the next section.

The next section, Sec. V, elaborates on the first steps of an ongoing program of numerical work aimed at understanding the interior of black hole-de Sitter spacetimes in more detail. The results and motivation of this first step, an investigation of the late time behavior of fields during gravitational collapse in de Sitter space, are presented. The conclusions of this study are discussed, with particular emphasis placed on their compatibility with the previous analytic studies of the interior.

Finally, we conclude in Sec. VI with a summary of the results presented in this contribution. In keeping with the mood of the Haifa meeting we also take this opportunity to speculate on the role that future work might take in the study of the Cauchy horizon in black hole-de Sitter spacetimes.

We also include two appendixes which provide additional material to the contribution, but need not be read in conjunction with the main body of text. An exhaustive bibliography is supplied at the end, which contains additional comments relating to each section.

Acknowledgments

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II. REISSNER-NORDSTRÖM-DE SITTER BLACK HOLE

A. The Spacetime

The generalization of the Reissner-Nordström black hole solution to include a cosmological constant has been provided by Carter [7]. In terms of advanced Eddington-Finkelstein coordinates (v, r) , the Reissner-Nordström-de Sitter metric is

$$ds^2 = -f(r)dv^2 + 2dvdr + r^2 d\Omega^2, \quad (1)$$

where

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{r^2}{\alpha^2}, \quad \alpha^2 = \frac{3}{\Lambda}. \quad (2)$$

In Eq. (2) M is the Bondi mass of the black hole, Q its electric charge and Λ is the cosmological constant. Throughout what follows, it will be assumed that Λ is a positive, non-zero, constant. The function $d\Omega^2 = d\theta^2 + \sin^2(\theta)d\varphi^2$ is the metric on the unit two sphere. The coordinate v is the standard advanced time coordinate, related to the Schwarzschild time coordinate t by

$$v = t + r_*, \quad (3)$$

with

$$r_* = \int \frac{dr}{f(r)} = -\frac{1}{2\kappa_1} \ln \left| \frac{r}{r_1} - 1 \right| + \frac{1}{2\kappa_2} \ln \left| \frac{r}{r_2} - 1 \right| - \frac{1}{2\kappa_3} \ln \left| \frac{r}{r_3} - 1 \right| + \frac{1}{2\kappa_4} \ln \left| \frac{r}{r_4} - 1 \right|, \quad (4)$$

where the arbitrary constant of integration has, for simplicity, been set to zero. The constants κ_j represent the surface gravity [8] at the corresponding j^{th} horizon, which are located at $r = r_j$. In general, the surface gravity is given by

$$\kappa_j = \lim_{r \rightarrow r_j} \sqrt{\frac{(\nabla_\mu |\ell^2|)(\nabla^\mu |\ell^2|)}{4|\ell^2|}}, \quad (5)$$

where ℓ is some suitably normalized Killing vector that is null on the j^{th} horizon and $\ell^2 = \ell_\mu \ell^\mu$. In the case of a static, spherically symmetric spacetime with a line element similar to Eq. (1) [9], then Eq. (5) reduces to

$$\kappa_j = \frac{1}{2} \left| \frac{df}{dr} \right|_{r=r_j}, \quad (6)$$

which can be verified using Eq. (1) and $\ell = \partial_v$. The location of the horizons r_j is given by the roots to the equation

$$f(r) \equiv 0, \quad (7)$$

which can be seen [via Eq. (2)] to reduce to a quartic equation in r . In general there will be four distinct roots [10] which we label r_1, r_2, r_3 and r_4 . The lack of a cubic term in Eq. (7) forces at least one of the roots to be negative and unphysical in this spacetime [11]. We order the roots as follows

$$r_1 > r_2 > r_3 > r_4 \quad \text{where} \quad r_4 < 0. \quad (8)$$

The root r_1 denotes the location of the *cosmological* event horizon, r_2 the location of the *black hole* event horizon and r_3 the location of the *inner* horizon (which from now on we shall refer to as the *Cauchy* horizon) of the black hole spacetime. We note that although r_4 does not correspond to a physical horizon in this spacetime, we are still able to define the constant κ_4 through Eq. (6). Though not strictly correct, κ_4 is often still referred to as a surface gravity. By factoring the metric function $f(r)$ as

$$f(r) = -\frac{(r - r_1)(r - r_2)(r - r_3)(r - r_4)}{\alpha^2 r^2}, \quad (9)$$

it is a simple task to show that the surface gravities are

$$\kappa_1 = \frac{(r_1 - r_2)(r_1 - r_3)(r_1 - r_4)}{2\alpha^2 r_1^2} \quad , \quad \kappa_2 = \frac{(r_1 - r_2)(r_2 - r_3)(r_2 - r_4)}{2\alpha^2 r_2^2}$$

$$\kappa_3 = \frac{(r_1 - r_3)(r_2 - r_3)(r_3 - r_4)}{2\alpha^2 r_3^2} \quad \text{and} \quad \kappa_4 = \frac{(r_1 - r_4)(r_2 - r_4)(r_3 - r_4)}{2\alpha^2 r_4^2} . \quad (10)$$

Using Eqs. (4), (9) and (10) it is not difficult to show that

$$\lim_{r \rightarrow r_j} f(r) = \pm 2(-1)^j r_j \kappa_j e^{2(-1)^j \kappa_j r_*} , \quad (11)$$

where (+) indicates that we approach the j^{th} horizon from above and the (−) from below.

B. Coordinate Systems

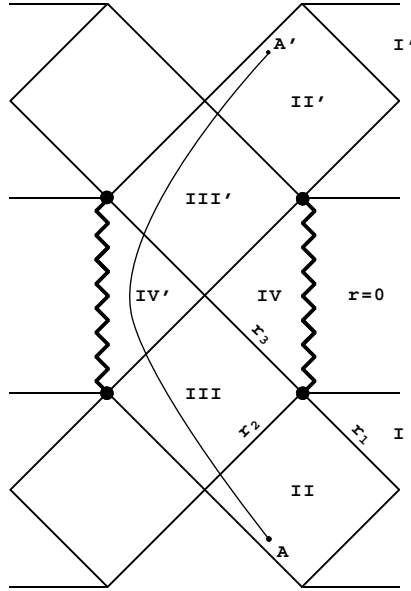


FIG. 1. A portion of the Penrose conformal diagram for the Reissner-Nordström-de Sitter black hole spacetime. The locations of the cosmological event horizon at $r = r_1$, the black hole event horizon at $r = r_2$ and the inner horizon at $r = r_3$, which is also a Cauchy horizon for external initial value problems, are shown. The dark wavy line represents the timelike spacetime singularity at $r = 0$. Also shown are the path of an infalling observer AA' , the four main spacetime regions I-IV and their isometric counterparts labelled I'-IV'. The diagram can be extended in all directions to reveal an infinite lattice of asymptotically de Sitter universes.

The Penrose conformal diagram [12] for the Reissner-Nordström-de Sitter black hole spacetime is shown in Fig. 1. As well as the location of the three horizons r_1, r_2 and r_3 , the timelike spacetime singularity at $r = 0$, and the path of radially infalling observer (see Sec. II C) AA' are also shown. It is clear from the conformal diagram that the horizons divide the spacetime into four main regions, labelled I-IV in Fig. 1,

$$\begin{aligned} \text{Region I} & : r_1 < r < \infty \\ \text{Region II} & : r_2 < r < r_1 \\ \text{Region III} & : r_3 < r < r_2 \\ \text{Region IV} & : 0 < r < r_3. \end{aligned}$$

Figure 1 also displays the primed regions I'-IV' which denote those regions of spacetime that are isometric to regions I-IV. In studying issues related to the Cauchy horizon, we are predominantly interested in region II, which we shall frequently refer to as the exterior, and region III, which we shall refer to as the interior. In physical terms, we can

consider region II as the *birth-site* for field fluctuations such as electromagnetic and gravitational waves [13]. These waves are continually scattered off the spacetime curvature, a fraction of which is transmitted across the cosmological event horizon in to region I, while the remainder is scattered through the black hole event horizon in to region III. The waves crossing the black hole event horizon eventually propagate throughout the interior region, perturbing the spacetime geometry there, and in particular the Cauchy horizon. It will, therefore, be beneficial if some time is devoted to each of these regions, paying special attention to the coordinate systems used there. For this purpose it proves useful to write the metric [Eq. (1)] in standard diagonal form using Eq. (3),

$$ds^2 = -f dt^2 + f^{-1} dr^2 + r^2 d\Omega^2. \quad (12)$$

Figure 2 shows enlarged versions of the regions II, and III, of Fig. 1. In each region, surfaces of constant r , the unbroken lines, and surfaces of constant t , broken lines, are shown. The corners of the regions are labelled A, B, C, D, E and F . These points can be seen to be singular points of the conformal diagram, in the sense that there is no one to one mapping between the corner points and the coordinates (t, r) . For the sake of clarity we shall consider each region individually.

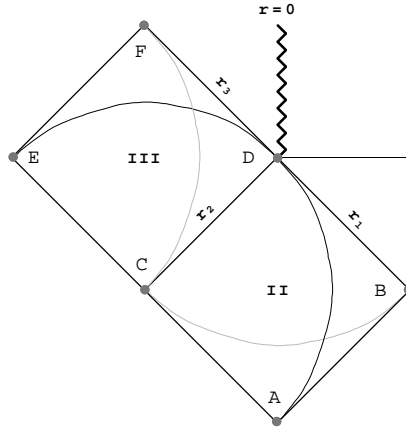


FIG. 2. The conformal diagram of the exterior and the interior, regions II and III respectively, of the Reissner-Nordström-de Sitter black hole spacetime. The unbroken curved lines represent surfaces of constant r whilst the broken curved lines represent surfaces of constant t . For orientation with Fig. 1 the horizons and singularity are also indicated.

Region II

In region II, depicted in Fig. 3, we define two radial, null coordinates u and v by

$$v = t + r_*, \quad (13)$$

$$u = t - r_*, \quad (14)$$

where v is defined exactly as in Eq. (3). In terms of (u, v) the spacetime metric [Eq. (1)] assumes the double null form

$$ds^2 = -f du dv + r^2 d\Omega^2, \quad (15)$$

whose form will be useful in Sec. VB, when we study scalar wave propagation on a Reissner-Nordström-de Sitter background. Surfaces of constant u and v are shown in Fig. 3 as lines at 45° to the horizontal, with $v = \text{const.}$ surfaces running parallel to the (future) cosmological event horizon BD and $u = \text{const.}$ surfaces running parallel to the (future) black hole event horizon CD . As indicated in the figure, we have defined the time coordinate t in Eq. (12) so that it is minus infinity on CAB and plus infinity on CDB . Using Eq. (4) one can determine that r_* takes the values minus infinity on ACD and plus infinity on ABD . With the definitions of u and v as above, [Eqs. (13) and (14)], it is not difficult to see that v assumes the values minus infinity on the (past) black hole event horizon AC and plus infinity on the (future) cosmological event horizon BD , whilst u is minus infinity on the (past) cosmological event horizon AB and plus infinity on the (future) black hole event horizon CD . In order to analytically continue the

spacetime across the boundaries AB, BD, CD and AC one has to introduce new coordinates that are regular there (unlike u and v which diverge). The new coordinates are furnished by the dimensionless Kruskal-Szekeres coordinates, conventionally labelled U and V . In later sections we shall be interested in both the (future) cosmological and black hole event horizons. For this reason we shall consider the (U, V) coordinates for the boundaries BD and CD only, the extension to the other boundaries being self-evident. Near CD we define

$$U = -e^{-\kappa_2 u} , \quad (16)$$

$$V = e^{\kappa_2 v} \quad (17)$$

so that U tends to zero as u tends to plus infinity. In terms of these two coordinates, the metric, Eq. (15), close to the (future) black hole event horizon becomes

$$ds^2 \simeq \frac{2r_2}{\kappa_2} dU dV + r^2 d\Omega^2 , \quad (18)$$

which is regular at the black hole event horizon. Similarly, close to BD , we define

$$U = e^{\kappa_1 u} , \quad (19)$$

$$V = -e^{-\kappa_1 v} \quad (20)$$

so that V tends to zero as v tends to plus infinity. Near to the (future) cosmological event horizon the metric in these coordinates becomes

$$ds^2 = \frac{2r_1}{\kappa_1} dU dV + r^2 d\Omega^2 , \quad (21)$$

which is regular at the cosmological event horizon.

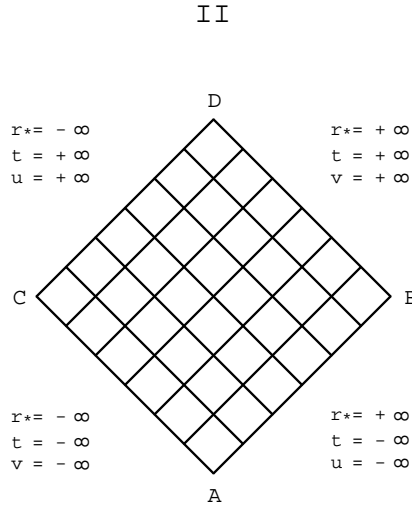


FIG. 3. Part of the conformal diagram of Fig. 1 showing region II and details of its causal structure (see the text for details).

Region III

Figure 4 depicts region III of Fig. 1. We define two radial null coordinates (u, v) as before, defined by Eqs.(13) and (14). Surfaces of constant u and v in Fig. 4 are shown as lines at 45° to the horizontal, with v running parallel to the Cauchy horizon DF and u running parallel to the black hole event horizon CD . It should be noted that the way we have defined u and v in region III is neither unique nor conventional [14]. Indeed, whilst r_* is constrained [via Eq. (4)] to take the values minus infinity along DCE and plus infinity along DFE , there is no such constraint on the parameter t . With u and v as defined, t assumes the values minus infinity along CEF and plus infinity along CDF ,

in order to preserve the requirement that both the Cauchy horizon and the cosmological event horizon be located at $v = +\infty$. One can equally well choose the more conventional definitions, $u = r_* + t$ and $v = r_* - t$, and have t be minus infinity on CDF and plus infinity on CEF [15]. No convention is strictly correct, but should be decided upon according to the problem at hand. The convention we choose is particularly well suited to the matching of waves across the event horizon, travelling from region II to region III. In looking at regular coordinate systems, near the boundaries of region III, we shall restrict our attention to DF , since the regular coordinates here play a crucial role when examining fluxes of energy crossing the Cauchy horizon, as we shall see in later sections. Near DF the regular coordinates (U, V) are defined by

$$U = e^{\kappa_3 u}, \quad (22)$$

$$V = -e^{-\kappa_3 v} \quad (23)$$

so that V tends to zero as v tends to plus infinity. The metric near DF , then becomes

$$ds^2 = -\frac{2r_3}{\kappa_3} dU dV + r^2 d\Omega^2, \quad (24)$$

which is obviously regular at the Cauchy horizon.

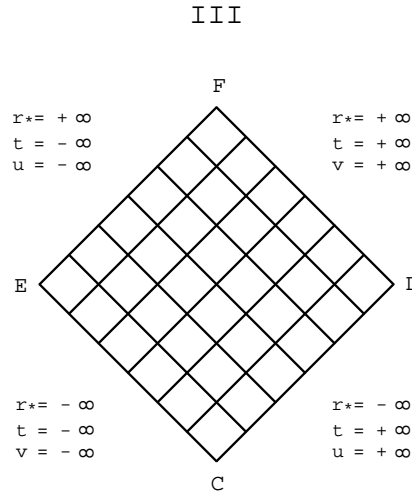


FIG. 4. Part of the conformal diagram of Fig. 1 showing region III and details of its causal structure (see the text for details).

C. Radial Geodesics

The role of geodesic observers, particularly what they measure, plays an important role in the study of any spacetime. In this section we shall consider certain concepts of geodesic observers that will be relevant to later sections. For the sake of both clarity and conciseness, we restrict our attention to radially free-falling observers ($d\theta = d\varphi = 0$). The first integrals of geodesic motion are

$$E^2 = \dot{r}^2 + f, \quad (25)$$

$$\dot{v} = \frac{E \pm \sqrt{(E^2 - f)}}{f} \quad (26)$$

where E is the constant of motion ('energy') associated with the timelike killing vector $\xi = \partial_v$, which is given by

$$E = -g_{\mu\nu} \xi^\mu u^\nu, \quad (27)$$

and $\mathbf{u} = d/d\tau$ is the observer's four velocity. The choice of sign in Eq. (26) depends upon whether or not the observer is travelling on a path of decreasing r , with respect to τ , in which case the negative root is taken, or whether they are

on a path of increasing r in which case the positive root is assumed. The path AA' in Fig. 1 represents the typical world line of such an observer. In region IV' the observer, having been on a path of decreasing r , reaches a turning point in her (or his) motion and proceeds to move along a path of increasing r . Eventually the observer reaches region II', another asymptotically de Sitter universe isometric to region II. The occurrence of a turning point in the evolution for $0 < r < r_3$ can be understood by examining the radial geodesic equation (25). If a turning point exists then $\dot{r} = 0$. The problem is then to ascertain at what, if any, value of $r > 0$ does this occur. Setting $\dot{r} = 0$ in Eq. (25) yields

$$f(r) - E^2 = 0, \quad (28)$$

the roots of which then define the turning points of the evolution. Figure 5 shows a plot of $f(r)$ vs. r for the generic case of three distinct spacetime horizons. The second plot is Eq. (28) vs. r , obtained by a vertical translation of $f(r)$ by an amount $-E^2$. For realistic observers there is one turning point in region IV'. Thus, there do exist paths for which radially infalling observers can enter the black hole, pass by the timelike spacetime singularity, and emerge in to another asymptotically de Sitter universe.

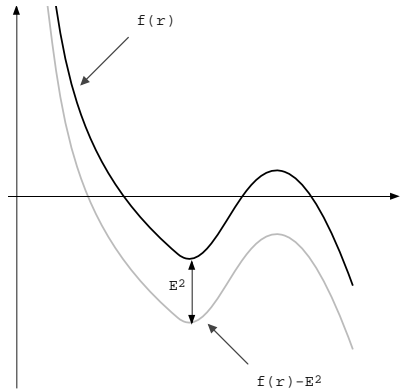


FIG. 5. A graph of $f(r)$ vs. r and $f(r) - E^2$ vs. r . The three zeros in $f(r)$ (from right to left) correspond to the spacetime horizons r_1, r_2 and r_3 . The zero in $f(r) - E^2$ at $0 < r < r_3$ corresponds to a turning point in the radial motion of a freely falling observer inside region IV', as shown in Fig. 1.

We now turn our attention to observers that are in the vicinity of either the cosmological event horizon or the Cauchy horizon. The importance of these observers, to the study of Cauchy horizon stability, will become transparent in later sections. For clarity in what comes later we use subscripts to denote which region we are considering. For radially free-falling observers approaching the cosmological event horizon, it is easy to show, using Eqs. (25) and (26), that

$$\dot{r}_{II} \simeq |E_{II}|, \quad (29)$$

$$\dot{v}_{II} \simeq \frac{2|E_{II}|}{f_{II}}, \quad (30)$$

The equation for \dot{r} merely tells us that our observer is approaching the future cosmological event horizon (increasing r) at a rate that depends upon his initial energy. The equation for \dot{v} is more interesting. As we approach the cosmological event horizon \dot{v} diverges, since f tends to zero there. From Eq. (11), with $r = r_1$, we can see that [16]

$$\dot{v}_{II} \equiv \frac{dv_{II}}{d\tau_{II}} \simeq \frac{|E_{II}|}{r_1 \kappa_1} e^{\kappa_1 v} \quad \text{as } v \rightarrow \infty. \quad (31)$$

For observers near the Cauchy horizon, it can similarly be shown that

$$\dot{r}_{III} \simeq -|E_{III}|, \quad (32)$$

$$\dot{v}_{III} \simeq -\frac{2|E_{III}|}{f_{III}}. \quad (33)$$

Again we can see that \dot{v} diverges as the observer approaches the Cauchy horizon, and one can show that

$$\dot{v}_{III} \equiv \frac{dv_{III}}{d\tau_{III}} \simeq \frac{|E_{III}|}{r_3 \kappa_3} e^{\kappa_3 v} \quad \text{as} \quad v \rightarrow \infty. \quad (34)$$

Since \dot{r} approaches a fixed constant value as we approach either horizon, the velocity of the observer will be dominated by the u^v component. Of course the divergence of \dot{v} is due purely to the choice of coordinates, and one can choose coordinates, such as the Kruskal-Szekeres coordinates of Sec II A, that are regular at a particular horizon and in which the four velocity is well behaved there.

III. AN HISTORICAL OVERVIEW

A. Mellor and Moss 1990

The first investigative study of Cauchy horizon stability in black hole-de Sitter spacetimes was performed by Felicity Mellor and Ian Moss in 1990 [17]. Confining their attention to the spherically symmetric Reissner-Nordström-de Sitter spacetime, Mellor and Moss considered the effect of gravitational perturbations, generated in the exterior of the black hole, on the Cauchy horizon in the interior. They examined the flux of radiation due to these perturbations, as seen by observers crossing the Cauchy horizon, and concluded that for the cases they studied, the Cauchy horizon is in fact stable to such perturbations [18].

Unfortunately a detailed exposition of the work of Mellor and Moss would lead to quite a *tour de force* in algebra and cause us to stray from picture we are trying to paint here. We will, therefore, sketch only an outline of their investigation.

Analysis

The method employed by Mellor and Moss is analogous to that used by Chandrasekhar and Hartle in their study of the Cauchy horizon instability in the Reissner-Nordström spacetime [19,20]. The technique is that of linear perturbations about the background spacetime

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + \epsilon h_{\mu\nu}, \quad (35)$$

where $\tilde{g}_{\mu\nu}$ represents the components of the perturbed metric, $g_{\mu\nu}$ the components of the background metric and $h_{\mu\nu}$ is the perturbation about the background. The dimensionless quantity ϵ is just an expansion parameter. By constraining the perturbed metric to be a solution to the Einstein-Maxwell equations (since there is a non-zero background electromagnetic field) to linear order in ϵ , one obtains linearized equations for the metric perturbations $h_{\mu\nu}$. These perturbations fall in to two distinct classes

Axial Perturbations : Change sign under a change of sign of the azimuthal angle φ

Polar Perturbations : Invariant under a change of sign of φ

With this definition it is easy to conclude, for a spherically symmetric background spacetime, that the polar perturbations have non-vanishing background values, whereas the axial perturbations vanish on the background. By considering the effect of the perturbations on the Ricci tensor, along with the linearized form of the Maxwell equations, it is quite remarkable that both the axial and polar perturbations can be cast in to the form of a set of one-dimensional Schrödinger type wave equations, four in all,

$$\frac{d^2 Z_j^\pm}{dr_*^2} + (\sigma^2 - V_j^\pm) Z_j^\pm = 0 \quad \text{for} \quad j = 1, 2. \quad (36)$$

Here the Z_j^- denote the two fields that describe the axial perturbations, Z_j^+ the polar perturbations and σ is the frequency of the perturbation modes with time dependence $e^{i\sigma t}$. The functions $V_j^\pm(r; \ell, M, Q, \Lambda)$ are the associated potentials, often referred to as the *effective potentials*, whose precise forms are too complicated to reproduce here but have been given by Mellor and Moss [17]. These equations are generalizations of the Regge-Wheeler and Zerilli equations, obtained from perturbations about asymptotically flat spacetimes [21]. One can, therefore, reduce the problem of linear perturbations to an ensemble of one dimensional scattering problems. In order to determine the whether or not the Cauchy horizon is stable, one examines the flux of radiation, due to the perturbations, encountered by an observer crossing the Cauchy horizon. The behavior of the flux near the Cauchy horizon is ultimately dictated

by the scattering the fields have undergone in propagating from the exterior to the interior, which is in turn governed by the transmission (T) and reflection (R) coefficients. Close to the Cauchy horizon, the modes $Z \in \{Z_j^\pm\}$ have the asymptotic form

$$Z \rightarrow A(\sigma)e^{-i\sigma r^*} + B(\sigma)e^{i\sigma r^*} \quad \text{as } r \rightarrow r_3. \quad (37)$$

As an observer approaches the Cauchy horizon, their four-velocity \mathbf{u} is proportional to ∂_V , where V is the Kruskal-Szekeres coordinate regular at the Cauchy horizon [see Sec. II B, Eq. (23)], and the flux of radiation he or she sees is proportional to

$$\partial_V z(t) = \frac{1}{\kappa_3} e^{\kappa_3 v} \partial_v z(t), \quad (38)$$

where $z(t)$ is the Fourier transform of the product of the mode function $Z(\sigma, t)$ and the initial data function $W(\sigma)$, namely

$$z(t) = \frac{1}{2\pi} \int W(\sigma) Z(\sigma, t) e^{i\sigma t} d\sigma. \quad (39)$$

With the asymptotic form for the mode function, Eq. (37), the flux near the Cauchy horizon is proportional to

$$\partial_V z(t) = -\frac{ie^{\kappa_3 v}}{2\pi\kappa_3} \int \sigma W(\sigma) A(\sigma) e^{-i\sigma v} d\sigma. \quad (40)$$

It should be noted that the more conventional definition for the null coordinates, u and v , described in Sec. II B have been used in obtaining Eq. (40). By implementing known results from one-dimensional scattering theory [19,21,22], and matching modes, scattered across the black hole event horizon from region II to region III, Mellor and Moss are able to show that

$$\overleftarrow{A}_{III}(\sigma) = \frac{\overleftarrow{T}_{II}(-\sigma)}{\overleftarrow{T}_{III}(-\sigma)} \quad \text{and} \quad \overrightarrow{A}_{III}(\sigma) = \frac{\overrightarrow{R}_{II}(-\sigma)}{\overrightarrow{T}_{III}(-\sigma)}, \quad (41)$$

where \overleftarrow{Z} denotes an ingoing mode, originating from the (past) cosmological event horizon (AB in Fig. 3) in region II and \overrightarrow{Z} denotes an outgoing mode, originating from the past black hole event horizon (AC in Fig. 3). The subscripts, of course, refer to the exterior and interior regions as defined in Sec. II B, and differ from those definitions in [17]. The analytic structure of the reciprocal transmission coefficients T_{II}^{-1} and T_{III}^{-1} are obtainable directly from the work of Chandrasekhar and Hartle [19]. Both T_{II}^{-1} and T_{III}^{-1} have poles along the imaginary σ axis at integer multiples of $i\kappa_+$ and $i\kappa_-$, where κ_- and κ_+ are the surface gravities (Sec. II A) of the horizons on the incident and transmission sides of the potential respectively. Due to zeros in T_{II} and R_{II} cancelling the pole in T_{III} , the leading pole in $A(\sigma)$ comes from a pole at $i\kappa_3$, whose residue results in a flux at the Cauchy horizon which is finite. The only other poles that could enter $A(\sigma)$ would come from T_{II} or R_{II} . Using numerical techniques, Mellor and Moss provided convincing evidence that there are no poles in either T_{II} or R_{II} in the range $0 < \text{Im}(\sigma) < \kappa_3$, thus concluding that the Cauchy horizon in a Reissner-Nordström-de Sitter black hole is stable to linear perturbations.

Remarks

There is one very subtle assumption in this analysis, which was only recognized later in the work of Brady and Poisson [23]. In studying the Reissner-Nordström-de Sitter interior, Mellor and Moss had not considered what constituted reasonable initial data in the exterior. We shall see that this is tantamount to ignoring the pole structure of the Fourier transform of the initial amplitude of the field modes $W(\sigma)$. By doing this there is an implicit assumption of vanishing flux at the cosmological event horizon which, on physical grounds, seems to be a rather strong restriction on the initial data. The more reasonable requirement of a finite, but non-zero, flux of energy at the cosmological event horizon still confirms that the Cauchy horizon can be stable, but only for a restricted family of solutions as we shall see next.

B. Brady and Poisson 1992

The question of Cauchy horizon stability in the Reissner-Nordström-de Sitter spacetime was revisited in 1992 by Patrick Brady and Eric Poisson [23]. Rather than contend with the formidable algebra of the gravitational perturbation method employed by Mellor and Moss, Brady and Poisson pursued a much simpler approach to the problem. Their model, a generalization of the study of perturbed a Reissner-Nordström interior due to Hiscock [24], mimics the perturbations propagating from region II to region III, across the event horizon, as a spherical inflow of null dust. Whilst an exact solution to the Einstein equations (in the presence of a cosmological constant) exists [25], Brady and Poisson instead treat the inflow as a linear perturbation, with the null dust propagating on a fixed Reissner-Nordström-de Sitter background. The beauty of the model is its simplicity together with its ability to capture the essential features of the Cauchy horizon stability issue. Whilst the model does not lend itself to a detailed investigation of the interior, it does incorporate the subtlety of the initial data problem missed in the Mellor-Moss analysis and allows us to make the following three, important, conclusions

- The infinite time compression effect, discussed in the introduction, is a *sufficient* requirement for an instability of the Cauchy horizon, but it is not a necessary requirement. A *necessary* condition is an infinite compression of the ratio of differential proper times.
- The requirement of a finite but non-zero flux of energy crossing the cosmological event horizon (initial data) leads to a significantly different stability condition than that obtained by Mellor and Moss. Namely the Cauchy horizon is stable provided that the surface gravity there is less than that at the cosmological event horizon,

$$\kappa_1 > \kappa_3 . \quad (42)$$

- If a backreaction calculation similar to that used by Poisson and Israel [26] to study the interior of Reissner-Nordström were performed, the model suggests the following conclusions
 1. If $\kappa_3 > 2\kappa_1$ then there would be a mass inflation type singularity.
 2. If $2\kappa_1 > \kappa_3 > \kappa_1$ then there would be a divergent flux at the Cauchy horizon but the internal mass function would approach some finite asymptotic value.

Analysis

The inflow of spherical null dust propagating on the fixed background, described by Eqs. (1) and (2), is characterized by the stress-energy tensor

$$T_{\mu\nu} = \frac{L(v)}{4\pi r^2} l_\mu l_\nu , \quad (43)$$

where $L(v)$ is the *luminosity* function and $l_\mu = -\partial_\mu v$ is a vector tangent to ingoing radial null geodesics. An observer crossing the inflow, with a four velocity \mathbf{u} , measures a flux of energy ρ given by

$$\rho \equiv T_{\mu\nu} u^\mu u^\nu . \quad (44)$$

A radially free-falling observer in the vicinity of the cosmological event horizon measures, therefore, a flux

$$\rho_{II} \simeq T_{vv} \dot{v}^2 = \frac{|E_{II}|^2}{4\pi r_1^2} L(v) e^{2\kappa_1 v} , \quad (45)$$

where we have used Eq. (31) of Sec. II B. On physical grounds, we expect the flux of energy at the cosmological event horizon to be finite and in general non-zero. This requires that

$$L(v) = K(v) e^{-2\kappa_1 v} , \quad (46)$$

such that

$$\lim_{v \rightarrow \infty} K(v) \equiv K_\infty \neq 0 . \quad (47)$$

Brady and Poisson refer to this later condition as the *minimal requirement*, which ensures a non-zero flux is measured by the observer at the cosmological event horizon.

A radially infalling observer in the proximity of the Cauchy horizon measures a flux of energy ρ_{III} , due to the inflow, given by

$$\begin{aligned}\rho_{III} &= \frac{|E_{III}|^2}{4\pi r_3^2} L(v) e^{2\kappa_3 v} \\ &= \frac{|E_{III}|^2}{4\pi r_3^2} K(v) e^{2(\kappa_3 - \kappa_1)v},\end{aligned}\tag{48}$$

using Eq. (46). Thus, at the Cauchy horizon, if $\kappa_3 > \kappa_1$ the energy density diverges, indicating an instability of the horizon [18]. On the otherhand, if $\kappa_1 > \kappa_3$ then the energy density is finite, suggesting the horizon is stable.

Remarks

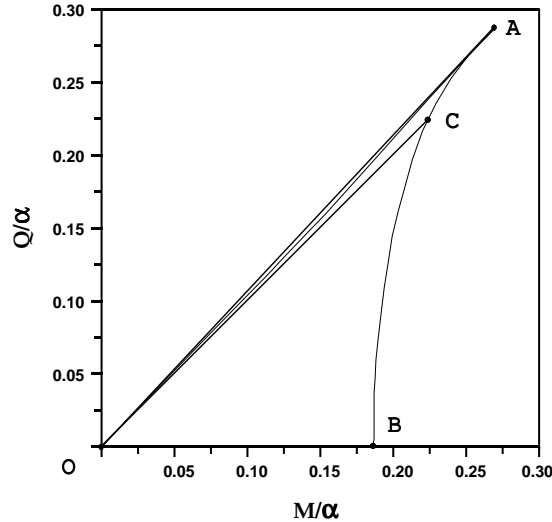


FIG. 6. A plot of the black hole parameter space showing the region of stability defined by Eq. (42). The upper line OA denotes the condition $r_2 = r_3$ whilst the line AB denotes the condition $r_1 = r_2$. The lower line OA, just visible, denotes the condition $\kappa_1 = \kappa_3$. The narrow region between the lines OA is the valley of stability. The line OC denotes the condition $|Q| = M$ showing that the stability condition requires $Q > M$ which can always be achieved by adding charge to the black hole.

The stability condition of Eq. (42) offered by the Brady-Poisson model is more restrictive than the original proposal of Mellor and Moss, and one might ask whether or not this condition can be met in any realistic sense. Figure 6 shows the region in the parameter space (M, Q, Λ) for which the stability condition holds. Though small, the *valley of stability* is of finite, non-zero, measure on the parameter space. An important point relating to this plot is that even if Λ is small, provided it is non-zero, one can always find solutions with a stable Cauchy horizon. There is absolutely no restriction to having an unacceptably large value of Λ [27]. Another point worth clarifying is that the stability condition requires $Q^2 > M^2$. Unlike the Reissner-Nordström black hole, there is no physical reason why one cannot keep putting charge on a Reissner-Nordström-de Sitter black hole to the point where the charge exceeds the mass [28]. In *practice* one could fine tune the black hole, by dropping in charge and/or mass, until it achieved a state lying in the stability region, allowing safe passage across the Cauchy horizon.

The stability condition also has the endearing quality that it can be understood intuitively, in terms of competing redshift and blueshift effects. At the cosmological event horizon there is an infinite redshift effect due to the cosmological expansion, whilst at the Cauchy horizon, due to its causal nature, there is an infinite blueshift effect. If the redshift outweighs the blueshift ($\kappa_1 > \kappa_3$) then the measured energy density at the Cauchy horizon is sufficiently diluted to render the horizon stable. Conversely, if the blueshift prevails over the redshift ($\kappa_3 > \kappa_1$) then the measured energy density is sufficiently dense to render the horizon unstable. Up until this juncture, the stability (or instability)

of the Cauchy horizon had always been explained in terms of the time compression effects between the exterior and interior, as discussed in the Introduction. The Brady-Poisson analysis demonstrates that while we can consider this a sufficient condition, it is not a necessary condition. For $\kappa_1 < \kappa_3$, the Cauchy horizon is unstable, but there is no infinite proper-time compression since the black hole is part of a de Sitter, i.e., closed, universe. What is necessary is an infinite compression of the ratio of *differential* proper-times, since blueshift and redshift are precisely differential proper-time effects (See Appendix A.)

The existence of a region in the parameter space for which the Cauchy horizon is unstable, but there is no associated growth in the internal mass function, lead Brady and Poisson to speculate that if backreaction were taken in to account [26], then the Cauchy horizon singularity might remain weak, in the sense that no scalars formed from the curvature tensors diverge there. Indeed, the Weyl scalar Ψ_2 , that in spherical symmetry is characterized by the mass function, is regular at the Cauchy horizon. However, we shall in Sec. IIID that there is in fact a scalar curvature singularity at the horizon, characterized by the divergence of the Kretschmann invariant $R_{\mu\nu\gamma\sigma}R^{\mu\nu\gamma\sigma}$.

C. Mellor and Moss 1992

In light of the work by Brady and Poisson, Mellor and Moss have reassessed their stability analysis [29]. By requiring that observers crossing the cosmological event horizon should measure a finite but non-vanishing flux of energy, it is relatively easy to show that the gravitational perturbation approach to the Cauchy horizon problem reproduces the stability condition, Eq. (42), in complete agreement with the Brady-Poisson model.

Analysis

The initial data requirement of a finite but non-vanishing flux of energy being measured at the Cauchy horizon is easily incorporated in to the gravitational perturbation method by imposing pole structure on the initial data function $W(\sigma)$, defined in Eq. (39). Close to the cosmological event horizon, the modes have the asymptotic form

$$Z \rightarrow e^{i\sigma r_*} + R(\sigma)e^{-i\sigma r_*}.$$

The flux of energy [Eq. (38)] near the cosmological event horizon is thus proportional to

$$\partial_V z(t) = \frac{ie^{\kappa_1 v}}{2\pi\kappa_1} \int \sigma W(\sigma) e^{i\sigma v} d\sigma, \quad (49)$$

Now, if $W(\sigma)$ has any poles in the range $0 < \sigma < i\kappa_1$, then the flux would diverge, indicating an unphysical instability of the cosmological event horizon. On the otherhand, if there are no poles in the range $0 < \sigma \leq i\kappa_1$ the flux is vanishing at the cosmological horizon. From a physical point of view this is a very strong requirement on the initial data. If we impose the more reasonable requirement of a finite, non-vanishing flux at the cosmological event horizon then $W(\sigma)$ is forced to have a pole at $\sigma = i\kappa_1$. Using the results of Sec. III A, this implies that the Cauchy horizon will be stable provided that $\kappa_1 > \kappa_3$, as initially suggested by Brady and Poisson.

Remarks

From the analysis above it is clear that neglecting the relevant pole structure of the initial data, encoded in $W(\sigma)$, imposes an unreasonably strong restriction on the initial perturbations, namely the flux [Eq. (49)] at the cosmological event horizon, due to these perturbations, vanish. In relation to the Brady-Poisson model, this condition would require

$$\lim_{v \rightarrow \infty} K(v) = 0,$$

which can be seen, via Eq. (48), to lead to stability in all cases and clearly explains the original result of Mellor and Moss.

D. Brady, Núñez and Sinha 1993

Unlike the pure inflow model of Brady and Poisson, the perturbative content of the analysis by Mellor and Moss does incorporate the effects of field-scattering off the spacetime curvature. However, neither analysis takes account of non-linear backreaction effects produced by the field evolution. The first, and so far only, calculation to consider the effects of backreaction on the geometry was provided by Brady, Núñez and Sinha [30] in 1993.

Generalizing the backreaction model devised by Poisson and Israel [26], for their study of the Reissner-Nordström interior, Brady *et al.* arrived at the following important conclusions

- The Cauchy horizon in Reissner-Nordström-de Sitter is stable provided that

$$\kappa_1 > \kappa_3 .$$

- For $2\kappa_1 > \kappa_3 > \kappa_1$
 1. There is a divergent flux of energy at the Cauchy horizon but no corresponding growth of the internal mass function.
 2. The curvature singularity at the Cauchy horizon is *strong*, revealing itself as a divergence of the Kretschmann scalar $R_{\mu\nu\gamma\sigma}R^{\mu\nu\gamma\sigma}$.
- For $\kappa_3 > 2\kappa_1$ the Cauchy horizon instability is qualitatively similar to that found in the Reissner-Nordström solution.

Analysis

The analysis of Brady *et al.*, whilst simple, does require a suitable knowledge of certain background material, which, for contextual reason, we have not presented here. In fact, the generalization of the Poisson-Israel model to include a cosmological constant is so straight forward that much of what one would learn from it can be gained from an understanding of the Poisson and Israel analysis [31]. For that reason we shall not attempt to reproduce an outline of the work by Brady *et al.* For the interested reader Appendix B details a simpler picture of the Brady-Núñez-Sinha backreaction model, originally due to Ori [32], which captures some of the essence of their analysis.

Remarks

The relevance of the work by Brady, Núñez and Sinha, to the study of Cauchy horizon stability in Reissner-Nordström-de Sitter spacetimes, cannot be over emphasized. As the only model, to date, to incorporate the effects of backreaction, it is both a natural and important extension to the preceding analyses. The agreement between the linear and non-linear studies is pleasing and, at present, is the best evidence we have toward the conjecture that the Cauchy horizon in black hole-de Sitter spacetimes is stable for $\kappa_1 > \kappa_3$. However, we should be careful not to paint too much of a rosy picture here. Two possible caveats to the work of Brady *et al.* that have direct bearing on the conjecture are

1. While the model includes backreaction, it does restrict these effects to a spherically symmetric spacetime.
2. The model imposes the minimal requirement, of Brady and Poisson, on the fields.

The first caveat is extremely difficult to address. To be more clear about this point, we have to ask what it is we are trying to prove in our studies of the Cauchy horizon in black hole-de Sitter spacetimes. We can pose this as a question;

Can generic collapse, in de Sitter space, lead to a black hole with a stable Cauchy horizon?

It is difficult, if not impossible, to answer this question. Currently our best approach is to investigate more tractable models of the interior, such as those restricted to spherical symmetry, to guide our understanding and hopefully one day answer our questions. To what extent the backreaction will play a role in the stability issue is unclear. Indeed, since the curvature appears to be regular at the Cauchy horizon (in the case of stability) we might expect that the backreaction plays a minor role in the shaping of the spacetime interior. However, it may turn out that in a fully

non-linear simulation of the collapse that the backreaction becomes important at early times, causing the interior to be significantly different to what our models predict. While we cannot provide definite answers to this and similar questions, we can gain some insight from our current models. With the conclusions of the linear model by Mellor and Moss in conjunction with the results of Brady *et al.* it is clear that the backreaction is not playing a significant role in the evolution of the spacetime close to the Cauchy horizon, at least within the confines of spherical symmetry and null dust flow. What about axisymmetric spacetimes? In Sec. III E we shall discuss the details of a recent perturbation analysis of the Kerr-de Sitter spacetime, similar to that implemented by Mellor and Moss in their study of the Reissner-Nordström-de Sitter spacetime. The results of this analysis suggest that rotation has no effect on the stability conjecture, within the perturbative approach employed. Unfortunately no backreaction calculation, akin to that of Brady *et al.*, currently exist for Kerr-de Sitter spacetimes, against which we could compare the linear analysis. We shall have to wait to see whether backreaction, away from the confines of spherical, in anyway alters our current picture of stability.

The second caveat is more readily addressed. The requirement of a finite but non-vanishing flux at the Cauchy horizon leads to Eqs. (46) and (47) for the luminosity function $L(v)$. There is little doubt that this is the most physically reasonable condition one could impose on the initial perturbations. For the case of black holes in asymptotically flat black hole spacetimes there is no such requirement on the luminosity function [33]. In fact, requiring the perturbations to vanish at future null infinity ($v = r = \infty$) just requires the luminosity to be a decreasing function of advanced time. The exact form for the luminosity function is arrived at from an analysis of the late time behavior of fields in the exterior due to Price [34] and Bičák [35]. For black holes in de Sitter space, the minimal requirement is implicitly imposing the form on the late time behavior of fields in the vicinity of the cosmological horizon. In Sec. V B we will discuss the details of a numerical investigation in to the late time behavior of fields in the Schwarzschild-de Sitter and Reissner-Nordström-de Sitter spacetimes and comment there on its compatibility with the minimal requirement of Brady and Poisson.

For $2\kappa_1 > \kappa_3 > \kappa_1$, the non-linear analysis shows the remarkable structure alluded to by Brady and Poisson [23] – a divergent flux but no corresponding mass inflation. In this instance, the Weyl scalar Ψ_2 no longer acts as a measure of the divergence as it does in the asymptotically flat case, instead it is the Kretschmann invariant that signals the existence of a scalar (strong) singularity at the Cauchy horizon. It had been assumed that the existence of a divergent flux at the Cauchy horizon would always be met by an associated growth in the internal mass parameter. That mass-inflation does not occur in this parameter range makes the black hole-de Sitter case all the more interesting and worthy of study.

E. Chambers and Moss 1994

Until 1994, all analyses of the Cauchy horizon in black hole-de Sitter spacetimes had, for simplicity, confined their attention to the spherically symmetric Reissner-Nordström-de Sitter solution. However, in general, one expects a black hole formed in a realistic gravitational collapse situation to be rotating and uncharged. It is therefore, both natural and physically well motivated to consider how the inclusion of rotation might affect the current spherical picture of Cauchy horizon stability. The generalization of the stability analysis by Mellor and Moss to the case of a rotating but uncharged black hole was performed in 1994 by Chris Chambers and Ian Moss [36]. Studying linear perturbations of scalar, electromagnetic and gravitational fields on the Kerr-de Sitter spacetime, Chambers and Moss were able to conclude that the Cauchy horizon was stable, provided that $\kappa_1 > \kappa_3$, as in the spherical case. The method employed by Chambers and Moss is identical to that used by Mellor and Moss in their study of the Reissner-Nordström-de Sitter spacetime (Sec. III A). The essence of the approach is to transform the equations for the perturbations to a set of one dimensional Schrödinger-type wave equations, thus reducing the problem of linear perturbations to the more familiar problem of one dimensional scattering. The entire analysis is, as in the Mellor and Moss study, a *tour de force* in algebra. Consequently we sketch only an outline of the analysis.

Analysis

Chambers and Moss consider three types of field perturbation

- Scalar perturbations (spin = 0)
- Electromagnetic perturbations (spin = 1)
- Gravitational perturbations (spin = 2)

The scalar perturbations are represented as a massless, minimally coupled scalar field obeying $\square\phi = 0$, propagating on a fixed background described by the Kerr-Newman-de Sitter spacetime. Using separable solutions of the form

$$\phi = R(r)S(\mu)e^{-i\omega t}e^{im\varphi} \quad (50)$$

where $\mu = a\cos(\theta)$ and a is the rotation parameter. Whilst no explicit solution to the angular equation exist, for $\Lambda = 0$ the solutions $S(\mu)$ are spheroidal wave functions. The equation for the radial function $R(r)$ is easily reduced to the form of a one dimensional scattering equation

$$\frac{d^2 Z}{dr_*^2} + VZ = 0, \quad (51)$$

where $Z(r) = (r^2 + a^2)^{1/2}R(r)$, r_* is the analogous coordinate to Eq. (4) for Kerr-Newman-de Sitter and V is the effective potential.

The electromagnetic perturbations on a Kerr-de Sitter background (no background electromagnetic field), are governed by Maxwell's equations. Chambers and Moss resorted to the Newman-Penrose (NP) formalism to describe these equations [37]. In the NP formalism, the six independent components of the field strength tensor $F_{\mu\nu}$, are described by the three complex Maxwell scalars (ϕ_0, ϕ_1, ϕ_2) . Trying separable solutions of the form Eq. (50), for each of the fields, enables one to write the equations for the radial functions as a set of one dimensional scattering equations similar to Eq. (51). The details of this transformation are quite involved and can be found in [39].

The gravitational field in the NP formalism is described by the five complex Weyl scalars $(\Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4)$, representing the ten degrees of freedom of the gravitational field. Due to the nature of the Kerr-de Sitter spacetime [40], Ψ_2 is the only non-vanishing Weyl scalar on the background and only the remaining Weyl scalars can be used to describe the gravitational perturbations. While Ψ_0 and Ψ_4 are gauge invariant, and hence measurable, Ψ_1 and Ψ_3 are not and, with a judicious choice of gauge, can be made to vanish. With a significant amount of algebra [41], the equations for Ψ_0 and Ψ_4 can, by assuming separable solutions of the form Eq. (50), be cast in the form of Eq. (51). Again, for the details, one is guided to [39].

With the equations governing the perturbations transformed to a set of one dimensional scattering equations, much of the hard work is over. An investigation of the flux of energy, due to the perturbations, near the Cauchy horizon is performed in an identical way to that implemented by Mellor and Moss. Imposing the minimal requirement of Brady and Poisson and examining the pole structure of the transmission and reflection coefficients, one can show [36] that the Cauchy horizon in Kerr-de Sitter is stable provided that, as in the spherical case,

$$\kappa_1 > \kappa_3,$$

Remarks

The analysis of Chambers and Moss is the first to indicate that Cauchy horizon stability, in black hole-de Sitter spacetimes, is not an artefact of spherical symmetry. Indeed, this analysis provides the best evidence yet that generic collapse in de Sitter space can yield black holes with stable Cauchy horizons, violating the spirit (if not the letter) of the strong cosmic censorship hypothesis. Unfortunately, there is currently no calculation that takes in to account the effects of backreaction on the spacetime. Thus we have no model against which we can test the predictions of the linear analysis. However, there is little evidence to suggest that a backreaction calculation would lead us to any other conclusions. The valley of stability in the Kerr-de Sitter case is similar to the charged case, shown in Fig. 6. One peculiar property of the Kerr-de Sitter spacetime is the existence of closed timelike curves, near the ring singularity at $r = 0$. These arise due to the timelike nature of the azimuthal angle φ for small values of r . The possibility of observers crossing in to this region (IV) faces us with the problem of doing physics in the presence of closed timelike curves. There are many interesting aspects of the Kerr-de Sitter spacetime which have not received attention but provide thought provoking possibilities [39].

F. Marković and Poisson 1995

While we are primarily concerned with the classical stability of the Cauchy horizon in black hole-de Sitter spacetimes, we would not be presenting a fair picture if we did not include the results of a quantum analysis.

Classically we have seen that the Cauchy horizon can be stable for a certain region of the black hole parameter space (M, Q, J, Λ) , which leads to problems associated with the loss of predictability that occurs beyond the Cauchy

horizon. It is natural, therefore, to ask whether or not the Cauchy horizon can be quantum mechanically stable or whether quantum effects will restore predictability. So, how do the fluxes of quantum fields affect the spacetime in the vicinity of the Cauchy horizon? The intriguing answer to this question was provided by Dragoljub Marković and Eric Poisson [42] in 1995. Examining the quantum fluxes measured by an observer approaching the Cauchy horizon, Marković and Poisson were able to conclude that the horizon is quantum mechanically unstable, except for the set of zero measure solutions with

$$\kappa_1 = \kappa_3 ,$$

which are represented by the lower line OA in the parameter space plot of Fig. 6.

Analysis

To investigate the quantum stability of the Cauchy horizon in a black hole spacetime generally requires a knowledge of $\langle T_{\mu\nu} \rangle$, the renormalized expectation value of the stress-energy tensor associated with the quantum field. However, even within the confines of spherical symmetry, four dimensional calculations of $\langle T_{\mu\nu} \rangle$ are extremely difficult and, for the case of black hole-de Sitter spacetimes, the situation is especially difficult because one cannot choose from the standard vacuum states [43], such as the Hartle-Hawking or Unruh states. Marković and Poisson instead consider a simpler, but still instructive, approach to the problem, quantizing a conformally invariant scalar field on a two dimensional version of the Reissner-Nordström-de Sitter spacetime

$$ds^2 = -f(r)dudv , \quad (52)$$

where the pair (u, v) are the null coordinates defined in Sec. II B and $f(r)$ is given by Eq. (2). A quantum state that is regular on both the cosmological horizons and the black hole event horizons, for the two-dimensional case, has been provided by Marković and Unruh [45]. For this state, the renormalized expectation value of the scalar field can be written as [46]

$$\langle T_{\mu\nu} \rangle = \theta_{\mu\nu} + t_{\mu\nu} + (48\pi)^{-1} R g_{\mu\nu} , \quad (53)$$

where R is the Ricci scalar associated with the two-dimensional metric [Eq. (52)], $\theta_{\mu\nu}$ is a state independent object and $t_{\mu\nu}$ is a state dependent object. Examining the expectation value of the energy density measured by a freely falling observer crossing the Cauchy horizon, $\langle \rho \rangle = \langle T_{\mu\nu} \rangle u^\mu u^\nu$, in a regular coordinate frame reveals that

$$\langle \rho_{III} \rangle = \frac{|E|^2}{48\pi} (\kappa_1^2 - \kappa_3^2) e^{2\kappa_3 v} , \quad (54)$$

as v tends to infinity [cf. Eq. (48)]. Thus, unless $\kappa_1 = \kappa_3$, the Cauchy horizon in Reissner-Nordström-de Sitter is quantum mechanically unstable.

Remarks

While the calculation is restricted to a simpler two-dimensional model, Marković and Poisson give a physical interpretation of Eq. (54), in terms of the thermal quanta emitted by the horizons and the gravitational redshifts and blueshifts such quanta undergo, that suggests the result should hold true in the four dimensional case and irrespectively of any particular quantum field.

The quantum mechanical instability of the Cauchy horizon is interesting for many reasons, but in particular it is that this situation demonstrates how, even in regions of spacetime where classical curvatures are not necessarily large, quantum effects can be important. One normally expects quantum effects to become important only when curvatures are sufficiently high (approaching planck scales). This result is similar to the situation in which a spacetime possesses regions containing closed timelike curves and regions that are free from them. The boundary between such regions, the *chronology* horizon, is sometimes stable classically but always quantum mechanically unstable. However, the quantum instability of the Cauchy horizon still standing, the existence of solutions to the classical Einstein equations with stable Cauchy horizons remains a disturbing issue.

IV. COMMENTS

In a classical setting, we can see that both linear and non-linear examinations of the interior of black hole-de Sitter spacetimes strongly suggest that the Cauchy horizon is stable for a finite, but non-zero, measure on the space of black hole parameters (M, Q, J, Λ) . Moreover, simple but effective studies of the stability issues [23] yield intuitively compelling, and pleasing, insights in to the mechanism at the heart of the Cauchy horizon stability condition, Eq. (42). Whilst the majority of researchers within the field are confident of the results and conclusions of the analyses in Sec. III, some doubts, concerning the validity of the linear perturbation studies, have been raised. It is worthwhile devoting some time to a discussion of these doubts, addressing them directly with the results and conclusions of Sec. III at hand.

In Sec. III A we briefly discussed the concept of a linear perturbation. For simplicity we shall consider a perturbation described by a scalar field. We can expand the scalar field as

$$\phi = \phi_0 + \sum_{n=1}^{\infty} \epsilon^n \phi_n, \quad (55)$$

where ϕ_0 is the background value of the field and ϕ^n represents the n -th order perturbation. Though not necessary here, we have introduced a dimensionless parameter ϵ which, in calculations, keeps track of the order of the perturbation. In the case of black holes the background field is, in general, zero. For linear perturbation studies one initially assumes that the scalar field is *sufficiently weak*, in the sense that the stress-energy associated with the field is negligibly small (no backreaction.) Formally this implies we consider the field equations to linear order in ϵ . The stress-energy tensor associated with the perturbation is quadratic in the field, and hence epsilon, so that to linear order the stress-energy is ignored. The spacetime is thus unaffected by the presence of the field and the field evolution takes place on a fixed background spacetime. If at any point of the evolution, the field becomes *large*, in the sense that its stress-energy becomes appreciable, then the linear theory is no longer valid and one must contemplate the construction of a model that takes account of non-linear effects. Indeed this is exactly what happens at the Cauchy horizon in the Reissner-Nordström spacetime [19]. The stress-energy tensor of the field becomes increasingly large as one approaches the Cauchy horizon. On the other hand, if the field remains *weak* everywhere in the region of interest then the linear theory, and its results, remain valid. As we discussed in the previous section, this is exactly what happens in the Reissner-Nordström-de Sitter spacetime. For $\kappa_1 > \kappa_3$ the fields stress-energy remains negligible and the results, conventionally, are taken to be representative of the full theory. Of course, for $\kappa_1 < \kappa_3$ the theory breaks down and is qualitatively similar to the case $\Lambda = 0$.

The doubts voiced about the results and conclusions of linear perturbation analyses concern themselves with precisely this conventional wisdom, that if a linear theory is finite then the full theory is finite too. Now, Eq. (55) involves an infinite sum over the field perturbations and although linear analyses show that ϕ_1 and its derivatives (appearing in the stress-tensor) are well behaved at the Cauchy horizon (for $\Lambda \neq 0$) this by no means guarantees that the infinite sum converges to a finite value [47]. In fact, not only should this sum be convergent of course but also those that appear in the stress-energy tensor. To prove that these sums converge to a finite value would be quite an undertaking, requiring a knowledge of the field behavior to all orders. A direct approach to the problem, showing convergence of the sums, is likely an impossible task. One possible approach, which would not directly prove convergence but could at least provide evidence of convergence, is based on an approach initially used by Ori to study non-linear perturbations in the Kerr spacetime [53]. This approach has been used to study the non-linear effects of a scalar field in the Reissner-Nordström-de Sitter spacetime [48], and predicts the correct linear behavior and allows one to obtain the behavior of each term in the expansion described by Eq. (55). By comparing the behavior of successive terms in the expansion, it may be possible to establish enough evidence to support the convergence of the sum. However, the work of Brady *et al.* [30] has, to some extent, answered this question already. Indeed, one can view this analysis as an alternative approach to addressing the convergence problem by directly attempting a non-linear study. That the results of this non-linear analysis agree with the results from the linear studies suggest that the conventional wisdom on linear perturbation theory is valid. This idea can even be taken one step further. Instead of assuming a model where null dust mimics the perturbations generated in the collapse, why not actually study the gravitational collapse of a body, which eventually forms a black hole? In this case one could attempt to follow the evolution of the collapse through the event horizon and in to the interior, paying special attention to spacetime near the Cauchy horizon. In the next section we shall address this idea in more detail.

V. CURRENT PROGRESS

One of the key ingredients to studying the interior of any black hole spacetime is a knowledge of the behavior of fields crossing the event horizon, due to scattering in the exterior of the black. In a realistic collapse situation these fields would always be present, as any initial perturbation (or any perturbation that forms during the collapse) in the collapsing body becomes dynamical after the onset of collapse and thus emits gravitational or electromagnetic waves. The waves, initially outgoing from the objects surface, will scatter due to their interaction with the spacetime curvature and a fraction of the wave will be reflected back to the surface. If an event horizon forms, there will always be a scattered component of the wave that crosses the event horizon and propagates through to the interior.

A. Radiative Tails

For black holes residing in asymptotically flat spacetimes, the behavior of these radiating fields at sufficiently late times after the collapse is well known now, both from analytic calculations [34,35] and from numerical studies of collapse [49,50]. The analytic and numerical studies both agree, that at late times ($v \rightarrow \infty$), along the event horizon, a physical field Ψ decays according to

$$\Psi \sim v^{-\gamma} \quad , \quad \gamma = 2\ell + 2 + P \quad (56)$$

where v is the standard advanced time coordinate of Sec. II A, ℓ is the multipole moment of the field ($\ell \geq s$) with spin s and P is a constant that assumes the value 0 if there is an initially static perturbation of the star and 1 otherwise. It is this late time decay that allows the gravitational field to relax to its final asymptotic state, parameterized by the three parameters associated with the non-radiatable multipoles ($\ell < s$) of the electromagnetic (Q) and gravitational fields (M, J) of the black hole. These late time fields are called the *tails* or *radiative tails* of collapse. Their importance lies in their role as initial data for studies of the interior, both analytically [26] and numerically [51,52]. In the analytic studies the late time behavior of the field is imposed as a restriction on the stress-energy tensor of the matter that is flowing in to the hole across the event horizon (i.e the field is supposed to mimic the late time component of the field scattered in to the black hole).

For pure inflow models we have seen that the stress-energy has the form given by Eq. (43). The energy density measured by a freely falling radial observer, crossing the Cauchy horizon in an asymptotically flat black hole, is [Eq. (44)]

$$\rho_{CH} = \frac{|E^2|}{4\pi r_{CH}^2} L(v) e^{2\kappa_{CH} v} \quad , \quad (57)$$

where to avoid confusion with the de Sitter case we have let $r = r_{CH}$ denote the location of the Cauchy horizon and κ_{CH} its surface gravity. Whereas in the de Sitter case the form of the luminosity function $L(v)$ followed the requirement of a finite but non-vanishing flux at the cosmological horizon, the situation in asymptotically flat spacetimes is different. The requirement that the flux as measured by an observer out at infinity approaches zero as v tends to infinity, just implies that $L(v)$ should tend to zero there. The form of $L(v)$ does not, therefore, follow naturally. To ascertain the form of $L(v)$ one examines the rate of change of the field, as measured by a freely falling observer

$$\dot{\Psi} \equiv \frac{d\Psi}{d\tau} = \Psi_{,\alpha} u^\alpha \quad , \quad (58)$$

where τ is the observer's proper time. Now, close to the Cauchy horizon the four-velocity of the observer is dominated by the u^v component, $\dot{\Psi}$. Therefore, as v tends to infinity

$$\dot{\Psi} \sim \Psi_{,v} \dot{v} \sim v^{-\gamma-1} e^{\kappa_{CH} v} \quad .$$

The flux ρ_{CH} is proportional to $(\dot{\Psi})^2$, from which we can deduce that

$$L(v) \sim \alpha v^{-2(\gamma+1)} \quad \text{as} \quad v \rightarrow \infty \quad , \quad (59)$$

where α is a constant.

To date, no analytic calculation of radiative tails in Reissner-Nordström-de Sitter or Schwarzschild-de Sitter exists, but it is surprising that the Brady-Poisson analysis did not actually require us to know any details of the form of the tails near either the black hole event horizon or the cosmological event horizon. Imposing a sufficiently general

requirement on the observed flux at the cosmological horizon (finite and non-zero) allows one to ascertain [See Sec. IIIB, Eqn. (46)] that

$$L(v) \sim K(v)e^{-2\kappa_1 v} , \quad (60)$$

where $K(v)$ is some slowly varying function of v that tends to a finite, non-zero value, as v tends to infinity. Reversing the argument above for this form of the luminosity function we get that

$$\Psi \sim e^{-\kappa_1 v} \quad (61)$$

at late times, near the cosmological event horizon. Thus, one would claim that the radiative tails in black hole-de Sitter spacetimes are exponential with a folding time given by the surface gravity at the cosmological event horizon.

B. Radiative Tails In Black Hole-de Sitter Spacetimes

We commented above that, unlike the case of black holes in asymptotically flat spacetimes, no analytic work on radiative tails in black hole-de Sitter spacetimes exists. This is either due to a lack of interest, or more likely the comparative difficulty of working in such spacetimes. Even in the case of Schwarzschild-de Sitter the analytic work rapidly becomes difficult and tedious. The majority of the difficulties can be traced to the fact that

$$g^{\alpha\beta}\nabla_\alpha r \nabla_\beta r = f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{r^2}{\alpha^2} \quad (62)$$

has four roots and as r tends to infinity

$$f(r) \rightarrow -r^2/\alpha^2 .$$

The lack of any detailed analysis, combined with the difficulties of attempting an analytic investigation, led Brady, Chambers, Krivan and Laguna [54] to perform the first numerical investigation in to the late time behavior of fields propagating in black hole-de Sitter spacetimes.

Brady, Chambers, Krivan and Laguna 1997

Brady *et al.* have studied, in some detail, the behavior of a massless, minimally coupled scalar field propagating on spherically symmetric spacetimes with a positive cosmological constant. Particular attention was focused on the late time behavior of the fields in three particular regions; (a) the cosmological event horizon, (b) the black hole event horizon and (c) future timelike infinity (point D in Fig. 3) – approached along surfaces of constant r between these two horizons. The methods they employ are similar to those used by Gundlach, Pullin and Price [49] in their numerical studies of radiative tails in asymptotically flat black hole spacetimes,

Linear Method The field propagates on the fixed background spacetimes of

- Schwarzschild-de Sitter
- Reissner-Nordström-de Sitter

Non-Linear Method The field is coupled to a general spherically symmetric spacetime through the Einstein-Klein-Gordon field equations

Linear Analysis

The idea behind a linear analysis is similar to that discussed in the considerations of linear perturbation theory. One assumes that the field is sufficiently weak that its effect upon the spacetime is negligible. This is tantamount to assuming that the scalar field is a linear perturbation on the spacetime, so that the stress-energy tensor which, for the field under attention, is

$$T_{\alpha\beta} = \phi_{,\alpha}\phi_{,\beta} - \frac{1}{2}g_{\alpha\beta}\phi_{,\gamma}\phi^{,\gamma} , \quad (63)$$

is second order in the field and thus vanishes to linear order. Linear analyses are not just favorable for their mathematical simplicity. In cases where the spacetime curvature is small, as in the exterior region of a black hole, the results from linear analyses are fairly representative of the results from non-linear analyses as we shall demonstrate later. Indeed Price's original work on tails in the Schwarzschild spacetime, a linear perturbation analysis, has been verified numerically both by linear and non-linear evolutions [49,50] to a high degree of accuracy.

In terms of the standard advanced and retarded times (u, v) the metrics for Schwarzschild-de Sitter and Reissner-Nordström-de Sitter are described by Eq. (15),

$$ds^2 = -f du dv + r^2 d\Omega^2 ,$$

with $f(r)$ given by Eq. (2), from which Schwarzschild-de Sitter is obtained simply by setting $Q = 0$. On this background, the massless minimally coupled scalar wave equation $\square\phi = 0$ becomes

$$\Psi_{,uv} = -\frac{1}{4}V_\ell(r)\Psi , \quad (64)$$

where the field has been decomposed in to spherical harmonics $\phi = \sum_{\ell,m} \Psi(u, v) Y_{\ell m}(\theta, \psi) r^{-1}$. The function $V_\ell(r)$ is the effective potential for the scalar field (Sec. III A) and has the following form

$$V_\ell(r) = f(r) \left(\frac{\ell(\ell+1)}{r^2} + \frac{f'(r)}{r^2} \right) , \quad (65)$$

where $'$ denotes derivatives with respect to the function's argument. Plots of the potential, between the cosmological and black hole event horizons, are given in Fig. 7.

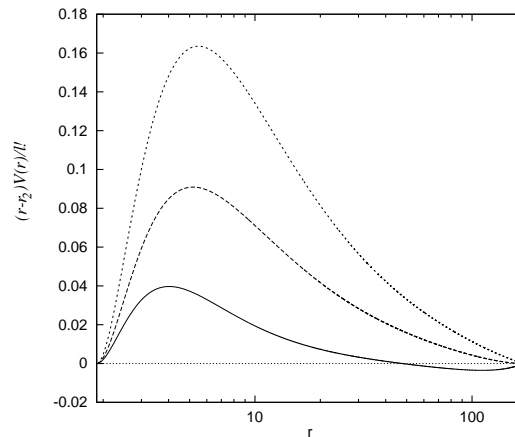


FIG. 7. The effective potential experienced by a scalar field propagating on the fixed background spacetime of a Reissner-Nordström-de Sitter black hole with $Q = 0.5$, $M = 1$ and $\Lambda = 10^{-4}$. Shown are the potentials for the $\ell = 0$ (solid), $\ell = 1$ (dashed) and the $\ell = 2$ (dotted) ℓ -pole moments. The potential has been scaled by $(r - r_2)/\ell!$ to accentuate the nature of the potential when $\ell = 0$. Unlike the higher ℓ modes, $V_{\ell=0}$ shows a barrier followed by a well. For any ℓ -pole moment, the potential falls off exponentially as r approaches either horizon.

Equation (64) is solved numerically by integrating it on the null grid defined by u and v . Initial data (characteristic) is given by the value of the field Ψ along two initial null surfaces $u = u_0$ and $v = v_0$. The details of the numerical method are adequately described in [49]. The results of the numerical integration are shown in Figs. 8, 9, 10 and 11.

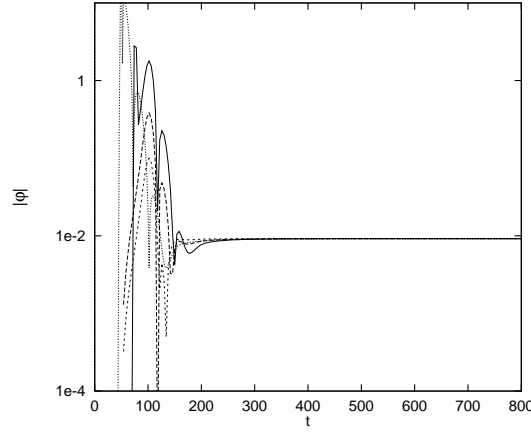


FIG. 8. A plot of $|\phi_{\ell=0}|$ versus time for a Schwarzschild-de Sitter black hole spacetime. The field behavior is shown along four different surfaces (see text).

The initial data used to generate these figures was

$$\Psi(u=0, v) = \exp \left[-\frac{(v - v_1)^2}{\sigma^2} \right]$$

$$\Psi(u, v=0) = \Psi(u=0, v=0),$$

being representative of the other data sets employed by Brady *et al.*. All the graphs shown are for black holes with their mass (M) scaled to unity and $\Lambda = 10^{-4}$. Though this choice of Λ is arbitrary the results are qualitatively similar for any other value of Λ that is non-zero. The fields are shown plotted along four different surfaces,

- The cosmological event horizon
- The black hole event horizon
- Two surfaces of constant r approaching future timelike infinity

Since the late time behavior is identical along each surface we shall not distinguish between them in the figures. Figure 8 displays the monopole field behavior for a Schwarzschild-de Sitter black hole. At early times ($0 < t < 200$) the field behavior is dominated by quasi-normal ringing, associated with complex characteristic frequencies of the hole. At late times $t > 200$ the field approaches the same constant value on all four surfaces.

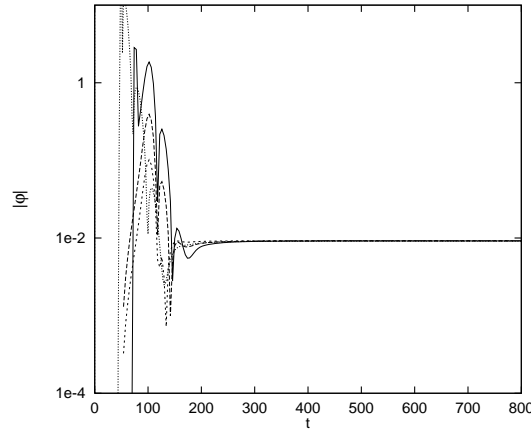


FIG. 9. A plot of $|\phi_{\ell=0}|$ for a Reissner-Nordström-de Sitter black hole with $Q = 0.5$. The field behavior is shown along four different surfaces (see text).

Figure 9 shows the same results for a Reissner-Nordström-de Sitter black hole with $Q = 0.5$. A period of quasi-normal ringing is followed by a relaxation of the field to a constant value on all four surfaces. A detailed investigation of the field's late time behavior reveals

$$\phi_{\ell=0} \simeq \phi_0 + \phi_1(r)e^{-2\kappa_1 t} , \quad (66)$$

and that the constant field term ϕ_0 scales like Λ [54].

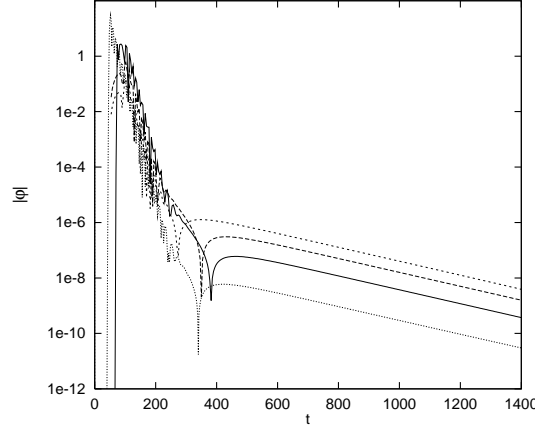


FIG. 10. A plot of $|\phi_{\ell=1}|$ for a Reissner-Nordström-de Sitter black hole for $Q = 0.5$. The field behavior is shown along four different surfaces (see text).

Figures 10 and 11 display the field behavior for $\ell = 1$ and the $\ell = 2$ modes for the case $Q = 0.5$. Again there is a period of quasi-normal ringing followed by a distinct exponential fall off. In general, an ℓ -pole mode of the field decays, at late times, like

$$\phi_\ell \sim e^{-\ell\kappa_1 t} \quad (\ell > 0) . \quad (67)$$

Non-Linear Evolution

If the scalar field is allowed to couple to the spacetime via the stress-energy tensor, then one must contend with the Einstein-Klein-Gordon field equations

$$G_{\alpha\beta} \equiv R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = 8\pi T_{\alpha\beta} - g_{\alpha\beta}\Lambda , \quad (68)$$

where $T_{\alpha\beta}$ is the stress-energy tensor for a massless, minimally coupled scalar field Eq. (63). For simplicity and tractability, attention is focused on spherically symmetric spacetimes, whose line element can be written as

$$ds^2 = -g\bar{g}du^2 - 2gdudr + r^2d\Omega^2 . \quad (69)$$

The Einstein field equations, Eq. (68), then reduce to

$$(\ln g)_{,r} = 4\pi r^{-1}(h - \bar{h})^2 , \quad (70)$$

$$(r\bar{g})_{,r} = g(1 - \Lambda r^2) , \quad (71)$$

$$(r\bar{h})_{,r} = h \quad (72)$$

and the scalar wave equation, $\square\phi = 0$, becomes

$$h_{,u} - \frac{\bar{g}}{2}h_{,r} = \frac{(h - \bar{h})}{2r} [g(1 - \Lambda r^2) - \bar{g}] , \quad (73)$$

where the auxiliary fields (h, \bar{h}) are defined by

$$\bar{h} = \frac{1}{r} \int h dr \equiv \phi. \quad (74)$$

Goldwirth and Piran [55] have devised a simple, but effective, numerical algorithm for integrating these equations on a (u, r) grid. This method has been implemented by Gundlach *et al.* to study radiative tails in asymptotically flat black hole spacetimes [49]. A refinement of this algorithm, which reduces numerical error near the $r = 0$ origin has been supplied by Garfinkle [56] and used by Brady *et al.* to integrate Eqs. (70)–(73). Initial data for the problem is given by the value of the field along some initial null cone centered at the origin of coordinates. Details of the numerical method can be found in [49] and references therein.

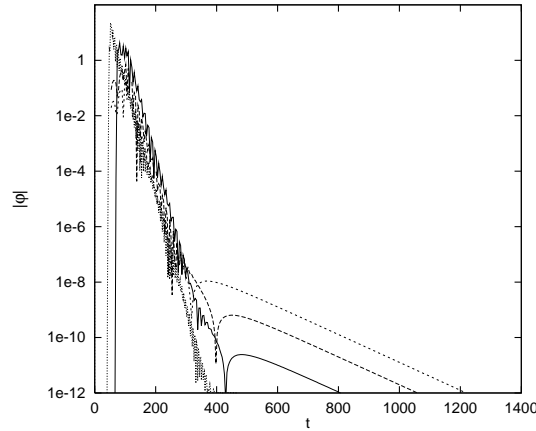


FIG. 11. A plot of $|\phi_{\ell=2}|$ for a Reissner-Nordström-de Sitter black hole for $Q = 0.5$. The behavior of the field is shown along four surfaces (see text).

In the graphs that follow, the initial data was Gaussian with

$$\phi = \phi_A \left(\frac{r}{r_0} \right)^2 \exp \left[-\frac{(r - r_0)^2}{\sigma^2} \right], \quad (75)$$

where ϕ_A is the amplitude of the field. The figures shown are, again, for $\Lambda = 10^{-4}$. The behavior of the field in this case is along three surfaces,

- The cosmological event horizon
- The black hole event horizon
- A surface of constant r approaching future timelike infinity

Again, because the qualitative behavior of the field at late times along each surface is similar, we do not explicitly distinguish each surface in the figures. The restriction to spherical symmetry implies that we gain only information about the $\ell = 0$ mode of the field, so the results plotted are for $\ell = 0$ only.

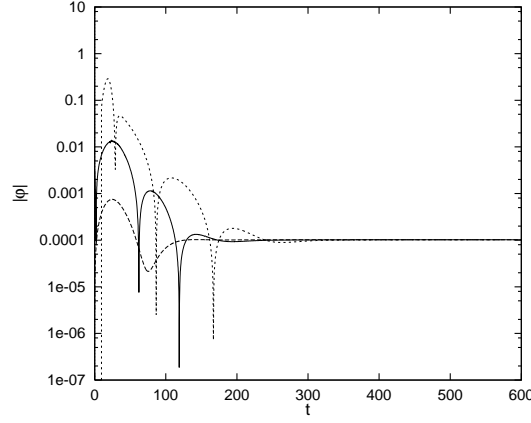


FIG. 12. A plot of $|\phi|$ versus time t for a non-linear evolution. The field behavior is plotted for three the different surfaces explained in the text

In Fig. 12 the field behavior can be seen to be remarkably similar to that demonstrated by the linear analysis. At late times the field approaches the same constant value along all three surfaces, in confirmation of the test field results.

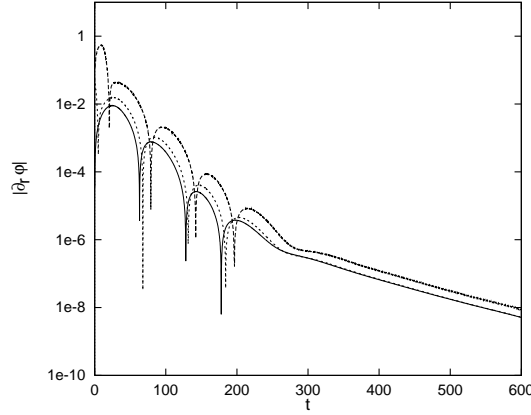


FIG. 13. A plot of the derivative of the field $|\phi_{,r}|$ versus time t for a non-linear evolution. The field behavior is plotted for three the different surfaces explained in the text

Figure 13 displays the behavior of $\phi_{,r}$, which is proportional to $(\bar{h} - h)$ along surfaces of constant r . Brady *et al.* find that at late times

$$(\bar{h} - h) \sim e^{-2\kappa_1 t} \quad \text{as } u \rightarrow \infty, \quad (76)$$

so that

$$\phi_{\ell=0} \simeq \phi_0 + \phi_1(r)e^{-2\kappa t} \quad \text{as } t \rightarrow \infty, \quad (77)$$

in agreement with the linear perturbation analysis.

C. Conclusions

For a massless, minimally coupled scalar field propagating on the fixed backgrounds of Schwarzschild-de Sitter and Reissner-Nordström-de Sitter, the late time behavior along the cosmological event horizon, the black hole event horizon and at future timelike infinity is given by

$$\phi \sim e^{-\ell\kappa_1 t},$$

for $\ell > 0$. For $\ell = 0$ behavior is slightly different,

$$\phi_{\ell=0} \sim \phi_0 + \phi_1 e^{-2\kappa_1 t},$$

asymptoting to a constant value at late times, which is ultimately confirmed by the non-linear evolution of a spherically symmetric scalar field coupled to general relativity.

The behavior of the $\ell = 0$ mode is, in some ways, unusual. As t tends to infinity, the field mode approaches a constant value, in fact the same constant value, at both the cosmological event horizon and the black hole event horizon. There is no analogue of this for the case of black holes in flat space. While Price's original work [34] demonstrates that there can be no static solutions to the scalar wave equation that are well behaved at infinity and the black hole event horizon, one can actually have the trivial static solution $\phi = \text{constant}$. The same holds true in black hole-de Sitter spacetimes. Even though the constant solution carries no stress-energy, it is intriguing that the appearance of a constant field value at the horizons does not occur in the flat space examples. An examination of the zero frequency reflection and transmission coefficients in the region exterior to a de Sitter black hole, shows they are non-zero, allowing the propagation of a constant mode to both horizons. In this respect, scattering in the exterior of a black hole-de Sitter spacetime is somewhat similar to scattering in the interior of a Reissner-Nordström black hole. However, in the exterior of a Reissner-Nordström black hole no such constant mode propagation is allowed. It is likely that the constant mode propagation in an $\ell = 0$ mode and the anomalous dip in the effective potential for that mode (Fig. 7) are not a mere coincidence, though no detailed investigation of this relation has been forthcoming. What is known is that for a conformally invariant scalar field the well in the potential disappears, and in this case the constant field is zero. A similar well in the effective potential occurs for the case of the gravitational perturbations detailed by Mellor and Moss [17]. It seems reasonable to suspect similar constant mode behavior will be observed in this case too [57].

At the start of this section on current work, we commented on how the minimal requirement of Brady-Poisson implicitly imposes form on the late time behavior of the perturbing field near the cosmological event horizon, described by Eq. (61). Since the minimal requirement is essential to the stability condition expressed in Eq. (42), it is important that the numerics support it. The numerical results have shown that we can, in general, expand the scalar field as

$$\phi(t, r) = \phi_0 + \sum_{n=1}^{n=\infty} \phi_n(r) e^{-n\kappa_1 t},$$

in the exterior at late times. Eq. (61) gives the form of the field we should expect from imposing the minimal requirement,

$$\phi \simeq a(r) + b(r) e^{-\kappa_1 t}.$$

We should note that this is a more general solution than that proposed in Eq. (61), where we dropped the constants of integration for simplicity. Making the identifications $a(r) = \phi_0$ and $b(r) = \phi_1(r)$ demonstrates that the numerical results confirm the minimal requirement of Brady and Poisson. It is worth noting that it is the $n = 1$, or $\ell = 1$ mode, that actually provides the required form for the radiative tails. The contribution from the other ℓ modes leads to a vanishing flux at the Cauchy horizon. In any generic perturbation we would expect all ℓ -pole moments to be present, thus assuring the correctness of the Brady-Poisson model.

VI. DISCUSSION

We began this review by stressing the need for a precise formulation and proof of the strong cosmic censorship hypothesis. It is almost twenty years since Penrose conjectured this stronger form of cosmic censorship [58], and today it still remains a cardinal, unsolved problem of general relativity. Instead of seeking a correct formulation of strong cosmic censorship, we have opted for a much simpler route – a search for reasonable counter-examples, in the belief that an understanding of these models will inevitably lead to a deeper comprehension of the censor issue. This hunt for counter-examples has lead us in to realm of black hole-de Sitter spacetimes and to pose the question

“Are black holes immersed in de Sitter space counter-examples to the current formulation of strong cosmic censorship?”

The answer to this appears, at least classically, to be yes. The linear perturbation analyses in Reissner-Nordström-de Sitter and Kerr-Newman-de Sitter and the backreaction calculations, performed in spherical symmetry, agree – the Cauchy horizon in black hole-de Sitter spacetimes is stable provided that

$$\kappa_1 > \kappa_3 .$$

Quantum mechanically the answer appears to be no. It wasn't obvious (to me at least) that this divergence couldn't just be due to a bad choice of vacuum state. To prove quantum instability one actually needs to prove there are no quantum states that are regular on all three horizons. Eric Poisson has kindly pointed out to me that although Marković and Poisson did not give this proof, one does exist, at least in the two dimensional case. The proof is provided in Eric's contribution elsewhere in these proceedings. It might be interesting to see if this results is easily continued to the four dimensional case. Occasionally, the results of a two dimensional calculation do not concur with the results of a four dimensional analysis. An excellent demonstration of this occurs in the extreme Reissner-Nordström spacetime. In two dimensions it has been shown [59] that stress-energy of a quantized field will diverge on the event horizon of an extreme black hole, whereas a four dimensional calculation shows no sign of a divergence [60]. It maybe that a four dimensional calculation of quantum effects near the Cauchy horizon will lead to a somewhat different conclusion than that reached by the two dimensionanl analysis. The quantum instability is undoubtedly an important question. However, the fiery marriage between general relativity and quantum mechanics, that is *Quantum-Gravity*, has yet to be consummated. For that reason we have concentrated mainly on the classical stability of the Cauchy horizon in black hole-de Sitter spacetimes. The existence of solutions to the classical field equations exhibiting stable Cauchy horizons faces classical physics with many problems, including a description of the singularity and how to attempt physics in the presence of causality violating curves.

A look to the Future

We now turn our attention to the future. The following comments are based purely on personal speculation about future work and the role it will play in answering the question posed above. The purpose of this section is not just to seek the truth, but to try to encourage a more active participation in the study of black hole-de Sitter spacetimes.

With the results of Brady *et al.*, on the form of radiative tails in Reissner-Nordström-de Sitter [54], things are pretty much set up to allow a numerical investigation of the interior. An adequate amount of numerical algorithms and techniques now exist, which have been used in the study of the Reissner-Nordström spacetime [50–52], and should allow this problem to be confronted with some ease. It is hoped that the numerical results will provide further evidence toward the conjecture that the Cauchy horizon in black hole-de Sitter spacetimes can be stable for a fixed, non-zero, region of its parameter space. A numerical study will, at the least, allow us to gain some insight in to the behavior of the spacetime in the vicinity of the Cauchy horizon and the behavior of fields propagating in the interior. These results can be used to gauge the adequacy of the linear analysis in describing the physics at the horizon and could provide information to initiate new analytic approaches for studying the interior.

A preliminary investigation of the numerical problem suggests that one should take an approach similar to that used by Brady and Smith [52] for studying the interior but implement the algorithm of Burko and Ori [52]. In this case the initial data is specified along some null surface that is taken to coincide with the event horizon. The alternative is to specify generic (Gaussian) initial data outside the hole (see [51,52]) and let it evolve, through the event, to the interior. For Reissner-Nordström this is quite adequate and has the advantage that one does not have to know the precise form of the radiative tails. For the de Sitter case this is not really a satisfactory approach. As we saw in Sec. VB, the restriction to spherical symmetry [Eq. (69)] implies that if we start from generic initial data outside the hole then at the event horizon we will have only the radiative tail of the $\ell = 0$ mode. The form of this tail, Eq. (77), does not satisfy the minimal requirement of Brady and Poisson discussed in Sec. III B, for this mode there is a vanishing flux of energy at the cosmological event horizon. To obtain a non-vanishing flux requires the introduction of higher ℓ -modes, specifically the $\ell = 1$ mode. Specifying the initial data along the event horizon, as Brady and Smith did, allows, in some respects, the added flexibility of incorporating the effects of additional ℓ -modes.

One of the biggest problems facing anyone attempting to study the interior of a Reissner-Nordström-de Sitter black hole is the search for an initial data set that evolves in to a solution with a stable Cauchy horizon. From Fig. 6, it is easy to see that the stability region is extremely small, in fact 0.6% of the entire parameter space by area. It might be possible, instead of hunting for initial data, to attempt some type of shooting method, with boundary conditions at the future event horizon and cosmological horizon. At present this is still a suggestion and may prove to be a dead end. Another approach which initially appeared sound was to fix regularity conditions, suggested by the analytic studies, on the spacetime and the fields at the Cauchy horizon and evolve them backward in time to the event horizon. The philosophy behind this idea being that one starts off with a solution that has a regular Cauchy

horizon in order to see whether or not it can evolve from regular data at the event horizon. Of course we expect the data corresponding to the field to have the exponential form required by the tails. The downfall of this approach is that derivatives of the field, at the Cauchy horizon, vanish. It seems likely therefore that the evolution will end up showing no field propagating through the interior as a whole, i.e., vanishing field perturbation at the event horizon. Currently this remains the most difficult step on the path toward a numerical investigation of the interior. While the feeling is that this is not an insurmountable problem, it seems it will require some thought.

APPENDIX A: DIFFERENTIAL PROPER TIMES

We present the arguments of Brady and Poisson [23] which demonstrate that a necessary condition for Cauchy horizon instability is that the ratio of *differential* proper times be divergent.

We consider two observers in the spacetime, an external observer in region II approaching the cosmological event horizon and an internal observer in region III approaching the Cauchy horizon. Each observer measures the proper time, τ_{II} and τ_{III} respectively, between two successive wavefronts, labelled v_0 and $v_0 + dv$. The situation is shown schematically in Fig. 14.

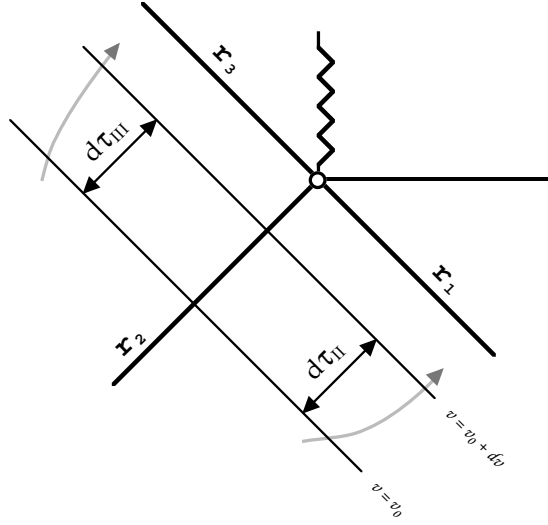


FIG. 14. A schematic representation of the blueshift vs. redshift argument of Poisson and Brady. Shown are the inner horizon (r_3), the outer horizon (r_1) and the black hole event horizon (r_2). Also shown are the paths of the two observers (light arrows) approaching their respective horizons. v_0 and $v_0 + dv$ represent two radially ingoing null rays, meant to represent two successive wave crests. The external observer measures a proper time $d\tau_{II}$ between the crests whilst the internal observer measures a proper time $d\tau_{III}$.

In region II, as the observer nears the cosmological event horizon, the u^v component of their four-velocity, \dot{v} , approaches

$$\dot{v}_{II} \equiv \frac{dv_{II}}{d\tau_{II}} \simeq \frac{|E_{II}|}{r_1 \kappa_1} e^{\kappa_1 v}, \quad (\text{A1})$$

where we have used the results of Sec. II B [Eq. (31)]. Similarly, for observers in region III approaching the Cauchy horizon,

$$\dot{v}_{III} \equiv \frac{dv_{III}}{d\tau_{III}} \simeq \frac{|E_{III}|}{r_3 \kappa_3} e^{\kappa_3 v}, \quad (\text{A2})$$

[See Eq. (34)]. Since the observers are measuring proper time between the same successive wavefronts ($dv_{II} = dv_{III} \equiv dv$) and because both horizons are located at the same advanced time, then

$$\frac{\dot{v}_{II}}{\dot{v}_{III}} = \frac{d\tau_{III}}{d\tau_{II}} \sim e^{(\kappa_1 - \kappa_3)v} \quad \text{as } v \rightarrow \infty. \quad (\text{A3})$$

Therefore, if $\kappa_3 > \kappa_1$, the interior observer measures an increasingly smaller proper time between successive wave crests as she/he approaches the Cauchy horizon indicating a blueshift and hence an instability of the Cauchy horizon. If, on the other hand $\kappa_1 > \kappa_3$, the observer measures an increasingly larger proper time between the wavefronts as he/she approaches the Cauchy horizon, indicating a redshift. In this case the Cauchy horizon is stable. Even without performing the calculation, it is not hard to see that these observers take a finite proper time to reach there respective horizons. One can also see the inevitability of the blueshift instability of the Cauchy horizon in the Reissner-Nordström solution. Here, the cosmological event horizon is replaced by by future null infinity, \mathcal{J}^+ , and $\kappa_1 \rightarrow 0$. In this case the internal observer will always measure an vanishingly small proper time between the wave crests as he approaches the inner horizon and so the Cauchy horizon is always unstable. This instability is due to the infinite time compression effects we discussed in the Introduction. Thus, we can conclude that while the infinite compression of the ratio of proper times is a sufficient condition for Cauchy horizon stability, it is not a necessary condition. A necessary condition is an infinite compression of the ratio of *differential* proper times expressed in Eq. (A3).

APPENDIX B: ORI-MODEL

The Poisson-Israel model of mass inflation [26] models the fluxes of radiation in the interior, generated during the collapse to form a black hole, as a crossflow of lightlike particles moving radially outward and inward respectively. Whilst the form of the inflow is crucial to the analysis (modelling the late time behavior of the fields crossing the event horizon) the nature of the outflux is largely irrelevant, its presence is required only to precipitate a contraction of the generators of the Cauchy horizon. In the Ori model the outflux is modelled as a thin lightlike shell Σ , which allows an exact mass inflation solution. The generalization of this model to black holes in de Sitter space is simple [30].

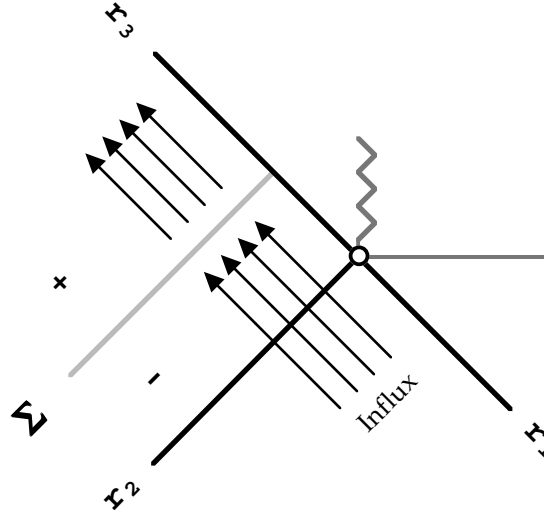


FIG. 15. The conformal diagram for the Ori model in the Reissner-Nordström-de Sitter spacetime. Shown are the locations of the cosmological event horizon r_1 , the black hole event horizon r_2 and the Cauchy horizon r_3 . Also shown is an influx of radiation crossing the event horizon (arrows) and the outflow of radiation, idealized as a thin lightlike shell Σ , emanating from the collapsing star's surface to the left of the diagram (not shown). The outflow naturally divides the spacetime up in to two distinct regions, that to its future (+) and that to its past (-).

Figure 15 depicts the conformal diagram of the situation. The null shell Σ divides the interior spacetime in to two distinct regions, that to the past of Σ which we label (-) and that to the future, which we label (+). On either side of the shell the spacetime is that of Reissner-Nordström-Vaidya-de Sitter (RNVDS), with metrics

$$ds_{\pm}^2 = -f_{\pm} dv_{\pm}^2 + 2dv_{\pm} dr_{\pm} + r_{\pm}^2 d\Omega_{\pm}^2 \quad (\text{B1})$$

$$f_{\pm} = 1 - \frac{2m_{\pm}(v_{\pm})}{r_{\pm}} + \frac{Q_{\pm}^2}{r_{\pm}^2} - \frac{r_{\pm}^2}{\alpha^2}. \quad (\text{B2})$$

The stress-energy tensor is

$$T_{\alpha\beta}^{\pm} = \frac{L(v_{\pm})}{4\pi r_{\pm}^2} (\partial_{\alpha} v_{\pm})(\partial_{\beta} v_{\pm}) . \quad (\text{B3})$$

The luminosity function $L(v)$ is related to the mass function $m(v)$ through the G_{vv} component of the Einstein equations

$$L(v) = \frac{dm(v)}{dv} \quad (\text{B4})$$

Matching

The problem is to ascertain the nature of the spacetime to the future of Σ , the $(+)$ region. This is simply done by matching the two RNVDs spacetimes across the shell. The first requirement is that metric tensor $g_{\alpha\beta}$ be continuous across Σ , which requires that $r_+ = r_- \equiv r$ and $d\Omega_+^2 = d\Omega_-^2 \equiv d\Omega^2$. We also impose the physical conditions that Σ be electrically neutral, so that $Q_+ = Q_- \equiv Q$ and that it be pressureless, which requires that $\lambda_+ = \lambda_- \equiv \lambda$, where λ is the affine parameter along Σ . The two important matching conditions that can be derived from these are

- The null generators of Σ are the same on either side

$$2dr = f_+ dv_+ = f_- dv_- \quad (\text{B5})$$

- Flux continuity across Σ

$$\frac{1}{f_+} \frac{dm_+}{d\lambda} = \frac{1}{f_-} \frac{dm_-}{d\lambda} \quad (\text{B6})$$

The first condition is given by the continuity of the metric tensor and that null geodesics satisfy $dr/dv = f/2$. The second condition is the requirement that $\lambda_+ = \lambda_-$, which can be shown [61] to be equivalent to the requirement that $T_{\alpha\beta}^+ m_+^{\alpha} m_+^{\beta} = T_{\alpha\beta}^- m_-^{\alpha} m_-^{\beta}$, where the m_{\pm}^{α} are the tangents to the null generators of Σ on either side. Thus the requirement of pressureless implies that Σ does not interact with the influx.

An Exact Solution

In the $(-)$ region we have that

$$L(v_-) = \frac{dm_-}{dv_-} = K(v_-) e^{-2\kappa_1 v_-} . \quad (\text{B7})$$

Integrating this gives

$$m_-(v_-) = M - \frac{\alpha}{2\kappa_1} e^{-2\kappa_1 v_-} , \quad (\text{B8})$$

where for convenience we have assumed that $K(v_-)$ is a constant α . In reality $K(v_-)$ is a slowly varying function of v_- and numerical analyses suggest that to first order $K(v_-)$ is constant. $M = m_-(\infty)$ is the final asymptotic mass. Equation (B5) gives that the null geodesics in region $(-)$ obey

$$\frac{dr}{dv_-} = \frac{f_-}{2} . \quad (\text{B9})$$

Expanding f_- about $r = r_3$ and letting v_- tend to infinity allows us to integrate this expression

$$\kappa_3(r - r_3) \simeq \beta e^{-\kappa_3 v_-} + \frac{\alpha \kappa_3}{2\kappa_1 r_3} \frac{e^{-2\kappa_1 v_-}}{(\kappa_3 - 2\kappa_1)} , \quad (\text{B10})$$

where β is a constant of integration. Then we can write the asymptotic form of f_-

$$f_-(r, v_-) \simeq -2\beta e^{-2\kappa_3 v_-} - \frac{2\alpha}{r_3(\kappa_3 - 2\kappa_1)} e^{-2\kappa_1 v_-} , \quad (\text{B11})$$

as $r \rightarrow r_3$ and $v_- \rightarrow \infty$. The second matching condition, Eq. (B7), allows us to obtain m_+ as a function of v_- ,

$$\int \frac{dm_+}{f_+} = \int \frac{dm_-}{f_-} = \int \frac{dm_-}{dv_-} \frac{1}{f_-} dv_- . \quad (\text{B12})$$

Using Eqs. (B2, B8) and (B11) it is easy to show that

$$m_+(v_-) \simeq M + \frac{\gamma\beta M(\kappa_3 - 2\kappa_1)}{\alpha} + \gamma M e^{(\kappa_3 - 2\kappa_1)v_-} , \quad (\text{B13})$$

where γ is another constant of integration. We can see immediately from this relation that the mass function to the future of the shell will inflate if and only if $\kappa_3 > 2\kappa_1$ which agrees with the work by Brady *et al.* [30]. We shall concentrate on the case that $\kappa_3 > 2\kappa_1$, the other cases follow the same line of reasoning. We can therefore estimate the leading behavior of the mass to be

$$m_+(v_-) \sim \gamma M e^{(\kappa_3 - 2\kappa_1)v_-} \quad \text{as} \quad v_- \rightarrow \infty . \quad (\text{B14})$$

We require to know how v_+ and v_- are related, which we can do by using Eq. (B5)

$$\int dv_+ = \int \frac{f_-}{f_+} dv_- . \quad (\text{B15})$$

Using Eqs. (B2, B11) and (B14) we find that

$$v_+ \sim -\frac{\alpha}{\gamma M(\kappa_3 - 2\kappa_1)\kappa_3} e^{-\kappa_3 v_-} , \quad (\text{B16})$$

where, without loss of generality, we have set the constant of integration to zero. If we remember that the regular coordinate at the Cauchy horizon is $V = -e^{-\kappa_3 v_-}$ [Eq. (23)] we can see that v_+ is Kruskal-like,

$$v_+ \sim \frac{\alpha}{\gamma M(\kappa_3 - 2\kappa_1)\kappa_3} V , \quad (\text{B17})$$

then

$$v_- \sim -\frac{1}{\kappa_3} | -V | . \quad (\text{B18})$$

The mass function $m_+(V)$ thus inflates as

$$m_+(V) \sim \gamma M (-V)^{\frac{2\kappa_1 - \kappa_3}{\kappa_3}} \quad \text{as} \quad V \rightarrow 0 . \quad (\text{B19})$$

We define a new coordinate U such that

$$-\delta dU = m_+ dv_+ + r dr \quad \text{where} \quad \delta = \frac{\gamma M(\kappa_3 - 2\kappa_1)r_3}{\alpha\kappa_3} , \quad (\text{B20})$$

so that close to the Cauchy horizon the metric becomes

$$ds_+^2 = -2e^{2\sigma} dU dV + r^2 d\Omega^2 \quad \text{where} \quad e^{2\sigma} = \frac{1}{\kappa_3^2} \frac{r_3}{r} \quad (\text{B21})$$

From Eq. (B20) we can obtain the behavior of the radius

$$r^2 = r_3^2 - 2U\delta + \frac{\alpha}{(\kappa_3 - 2\kappa_1)\kappa_1} (-V)^{2\frac{\kappa_1}{\kappa_3}} , \quad (\text{B22})$$

which reflects the slow contraction of the Cauchy horizon.

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- [16] One may be surprised to see the argument of the exponential in Eq. (31) contains the advanced time coordinate v rather than the radial coordinate r_* . However, along a radial geodesic that crosses either horizon, it can be seen [from Eqs. (29) and (30) or Eqs. (32) and (33)] that $dr/dv = \frac{1}{2}f$. Integrating this, and using Eq. (4), one can easily show that $r_* \sim \frac{1}{2}v + k$ as the horizons are approached. In Sec. IIC we have, without loss of generality, set the integration constant k to zero.
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